Calculation of Kolmogorov Entropy in Cavity Quantum Electrodynamics

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ABSTRACT— In this paper Kolomogorov entropy of a simulated cavity quantum electrodynamics in a multi-partite system consisting of eight quantum dots in interaction with one cavity mode has been estimated. It has been shown that the Kolmogorov Entropy monotonically increases with the increasing coupling strength, which is a sufficient condition for chaotic behavior under ultrastrong coupling regime. The arrangement of the quantum dots is assumed to be in the form of a linear chain where dipole-dipole interactions are considered only between the nearest neighbors.

KEYWORDS: Chaos, Kolmogorov Entropy, Cavity Quantum Electrodynamics, Quantum Optics.

I. INTRODUCTION

Entanglement has recently evolved to a major research topic in the field of theoretical and experimental Cavity Quantum Electrodynamics (CQED), because of its various potential applications in quantum computing, quantum information, and quantum communications. CQED describes the physical interactions between the quantum optical systems and confined light in the cavities [1]. Entanglement in multi-partite systems, for instance, has been shown its usefulness for switching between linear optical networks [2]. Therefore, a through study of this quantum phenomena is essential for getting insight into the fundamental design and operation of future entanglement-based quantum devices.

Recently, some advances have been made in the study of multi-partite quantum systems. In [3], for example, an exact description of electronic wave functions in solids has been proposed, which had been otherwise supposed to be impossible in the past. But, to the best knowledge of the authors, the most complete simulation of multi-partite quantum systems has been achieved on a physical diamond substrate [4], which requires sophisticated fabrication and measurement technologies. Hence, numerical simulations are essential.

In a new work of the authors [5], we have succeeded in devising an elaborate numerical simulation package, capable of simulating arbitrarily complex CQED systems. This work has been based on direct solution of the Jaynes-Cummings-Paul (JCP) Hamiltonian [6,7], using an extension devised in [8] and corrections to the Hamiltonian in [9,10].

We furthermore have successfully simulated a real three-level CQED quantum optical system designed in an earlier work [11], in interaction with one cavity mode. Also, simulation of a more complex quantum system with six quantum dots [5] and nine quantum dots [12] have been successfully studied by using this software under different coupling regimes.

We have noticed that under the so-called ultrastrong coupling regime, where the typical Rabi frequencies are comparable to, or at least of the order of, the atomic transition frequencies, a chaotic behavior starts to develop in all partitions of the multi-partite quantum systems.

Quantum chaos has been so far observed in optical [13], microwave [14], and nano-photonic [15] quantum systems. However, little information exists on how to rigorously calculate the measure of chaotic behavior in such systems.

Here, we show that the Kolmogorov entropy, denoted by K_2, is a fairly convenient measure [16] to calculate the quantum chaos in systems such as multi-partite CQED. We present the discussion by simulating a nine-partition chain-like CQED system, comprising of one cavity mode in interaction with eight light emitters.

II. SIMULATION

In this section, we describe the simulated system based on the theoretical formulations devised in [5, 12]. As depicted in Fig. 1, eight identical two-level light emitters are allowed to mutually interact in a linear-chain configuration through the electrostatic dipole-dipole interaction, which is limited to the nearest neighbors. One photonic mode, which may be occupied with a maximum number of eight photons, constitutes the field-emitter interactions. Such an apparently small quantum system, indeed would need a huge computation time because the burden of exact quantum computations explodes with the number of system partitions.

The most general state-ket of the system following the formulation in [8] is given by:

$$|\varphi(t)\rangle = \sum_{r_{1\ldots7},r_8=e,g}\sum_{f=0}^{8}\phi(A,f)|A\rangle|f\rangle$$
$$|A\rangle = \bigotimes_{n=1}^{8}\left|\begin{matrix}n\\ r_n\end{matrix}\right\rangle = \left|\begin{matrix}1\\ r_1\end{matrix}\right\rangle\left|\begin{matrix}2\\ r_2\end{matrix}\right\rangle\ldots\left|\begin{matrix}8\\ r_8\end{matrix}\right\rangle \qquad 1\le r_n\le 2 \tag{1}$$

where $|A\rangle$ is the state eigenket describing all possible combinations of the field emitters in their various eigenstates. r_n represents the energy level at which the n-th emitter stays. Also, $|f\rangle$ is the field eigenket of cavity modes. For practical reasons in the available simulation hardware, we limited the number of cavity modes only to one as stated in the above, with a maximum occupancy number of eight photons [5]. The corresponding Hamiltonian is then given by [5]:

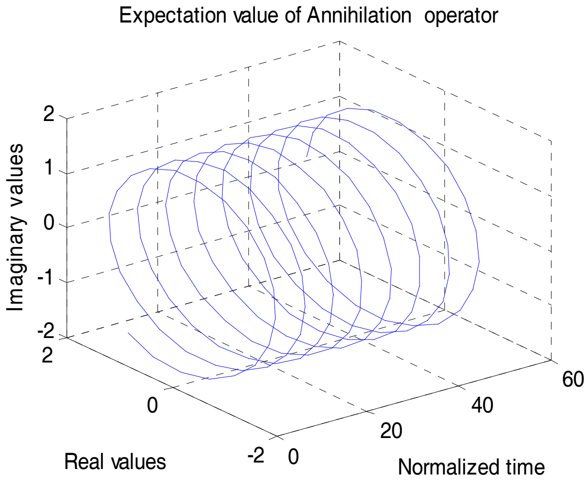
$$\begin{aligned}
\hat{\mathbb{H}} &= \hat{\mathbb{H}}_0 + \hat{\mathbb{H}}_{\mathbf{r\cdot r}} + \hat{\mathbb{H}}_{\mathbf{r\cdot E}}\\
\hat{\mathbb{H}}_0 &= \sum E_i^n \hat{\sigma}_i^n + \hbar\Omega\hat{a}^\dagger\hat{a}\\
\hat{\mathbb{H}}_{\mathbf{r\cdot r}} &= \sum_{n<m,i<j}(\eta_{nij}\hat{\sigma}_{i,j}^n + \eta^*_{nij}\hat{\sigma}_{j,i}^n)\left(\eta_{mij}\hat{\sigma}_{i,j}^m + \eta^*_{mij}\hat{\sigma}_{j,i}^m\right)\\
\hat{\mathbb{H}}_{\mathbf{r\cdot E}} &= \sum_{n,i<j}(\gamma_{nij}\hat{\sigma}_{i,j}^n + \gamma^*_{nij}\hat{\sigma}_{j,i}^n)(g_{nij}\hat{a} + g^*_{nij}\hat{a}^\dagger)
\end{aligned} \tag{2}$$

where $\hat{\mathbb{H}}_0$ is the basic Hamiltonian with no interaction, $\hat{\mathbb{H}}_{\mathbf{r\cdot E}}$ is the field-emitter interaction term, and $\hat{\mathbb{H}}_{\mathbf{r\cdot r}}$ is the dipole-dipole interaction terms. γ_{nij} are matrix elements of the dipoles corresponding to the n-th emitter, and g_{nij} are the interaction strengths between the optical field and the n-th emitter via the transition between i-th and j-th states. η_{nij} are proportional to the induced dipole terms while transiting between the i-th and j-th states. Finally, E_i^n are the energies of the i-th state for the n-th emitter. Selected numerical values are enlisted elsewhere [12].

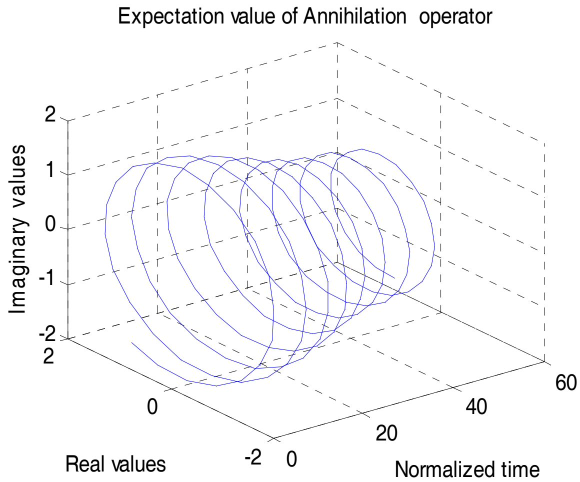
Fig. 1. Arrangement of quantum dots in the linear chain fashion [5].

Since, we aimed to study the effect of interaction strengths on the chaotic behavior of the system, we repeated the same simulation for various choices of g_{nij}. Now, employing the code [17] developed based on the algorithm discussed in [5,12], we may flatten a multi-dimensional state matrix into a two-dimensional square matrix, which subsequently allows calculation of its corresponding eigenvalues and temporal solutions of eigenkets.

We have calculated the occupancy probabilities at various emitter and field modes for the whole system. However, for the sake of brevity we limit the discussion to the expectation values of the two photon annihilator and atomic transition operators.



(a)



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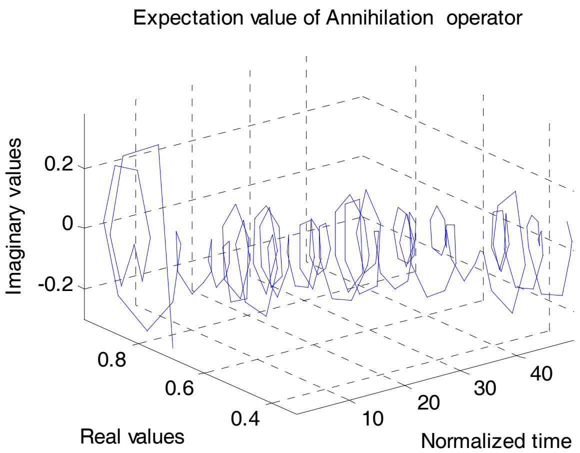
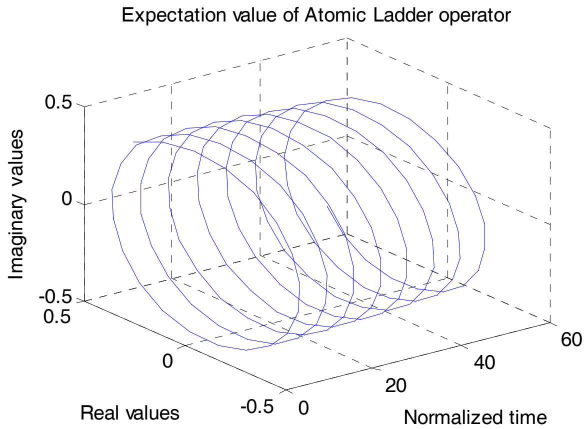


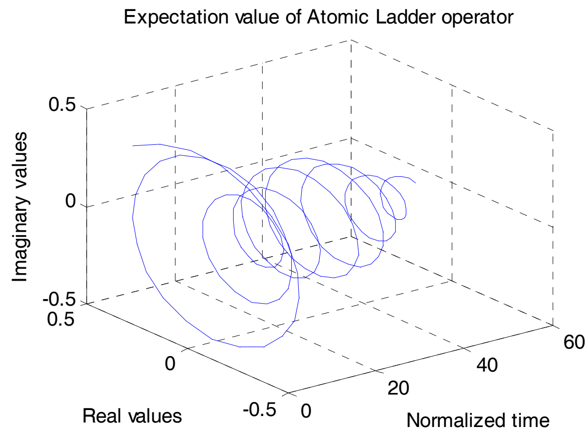
(c)

Fig. 2. Expectation values of the Annihilator versus time, (a) weak; (b) strong; (c) ultrastrong coupling [12].

It has to be here mentioned that both of these two operators are non-Hermitian and their expectation values in general yield complex numbers, which have no direct physical significance. It may not be customary in the standard literature of quantum mechanics to calculate such non-Hermitian products.



(a)



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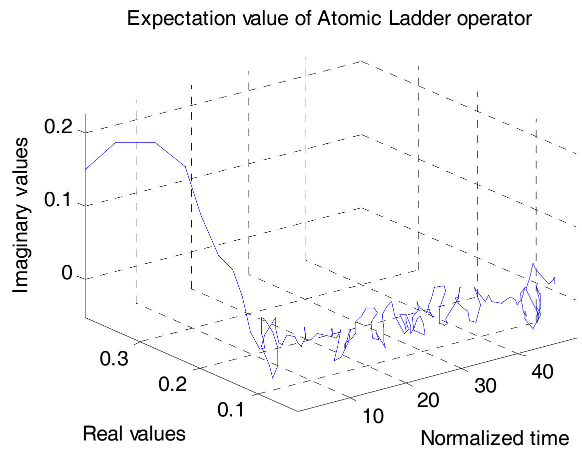

(c)

Fig. 3. Expectation values of the atomic ladder versus time: (a) weak; (b) strong; (c) ultrastrong coupling. The general behavior of atomic and field expectations are identical under various coupling regimes [12].

However, this is exactly what the authors are looking for: to successfully define and apply an easily-computable scalar variable summarizing all internal phase information of a multi-partite quantum system. Such phase information are preserved only in complex-valued calculations and are normally missed in real-valued Hermitian expectations. However, chaos is very evident and easily detectable in the phase information, rather than the magnitudes.

These expectations for the field annihilator and transition operator of the second emitter are given by

$$\left\langle \varphi(t)|\hat{a}_{\nu}|\varphi(t)\right\rangle = \sum_{r_{1\ldots7},r_8=g,e}\sum_{f_1=0}^{8}\sqrt{f_1}\,\phi^*(r_1,\cdots,r_8,f_1-1)\times\phi(r_1,\cdots,r_8,f_1) \quad (3)$$

$$\left\langle \varphi(t)|\hat{\sigma}^2_{g,e}|\varphi(t)\right\rangle = \sum_{r_{1,3,4,5,6,7,8}=g,e}\sum_{f_1=0}^{8}\phi^*(r_1,r_2\rightarrow g,r_3,\cdots,r_8,f_1)\times(r_1,r_2\rightarrow e,r_3,\cdots,r_8,f_1) \quad (4)$$

in which $\hat{a}_{\nu}$ is the field annihilator and $\hat{\sigma}^2_{g,e}$ is the excitation operator belonging to the second emitter.

Figs. 2 and 3 illustrate these values (3, 4) as a function of time for various coupling strength. As it is evident, by entering into the stronger interacting, chaotic behavior starts to develop. We have identified the same behavior in all studied examples so far [5, 12], too. We also extended the dipole-dipole coupling to the second and the third neighbors (albeit with somewhat diluted strength) and found essentially no difference in these observations.

III. Calculation of K_2 Entropy

After computation of the expectation values, we obtain a complex time-signal for every non-Hermitian product, such as (3, 4). One would need a method to mathematically show whether the resulting signal is truly chaotic or not.

The perfect solution to this question lies within calculation of the second-order Kolmogorov entropy, denoted by K_2. As it has been shown in [16] for a given time-signal, a non-zero K_2 is a sufficient (and not necessary) condition for existence of chaos. While K_2 is zero for fully deterministic signals, it tends to infinity for the white noise. The K_2 entropy is defined as

$$K_{2,d}(\varepsilon) = \frac{1}{\tau}\ln\frac{C_d(\varepsilon)}{C_{d+1}(\varepsilon)} \quad (5)$$

in which τ represents the duration of time-intervals, and is set to 0.1 in our calculations for the normalized time-step. Furthermore, we have

$$C_d(\varepsilon) = \lim_{N\to\infty}\frac{1}{N^2}Num\left\{\left[\sum_{i=1}^{d}\left|X_{n+i}-X_{m+i}\right|^2\right]^{1/2} < \varepsilon\right\} \quad (6)$$

where N is the length of sample in the signal X_j, and $Num\{\cdot\}$ represents the number of arguments satisfying the condition given therein.

In our simulation, N=1500 and $\varepsilon=10^{-M}$ was set to values ranging from M=0 all the way to M=7. The K_2 entropy as defined by (5) converges to some non-negative real value by increasing M, that is smaller values of the infinitesimal parameter ε. Because of the particular way in which K_2 entropy is defined, the signal X_j, being here either of the non-Hermitian products (3, 4) may be normalized to attain values falling within the unit imaginary circle. This will help expedite the calculation. We developed MATLAB codes for calculation of K_2 entropy. Illustrated in Fig. 4, the algorithm is much simpler than what it seems at first.

The K_2 entropy was calculated for the non-Hermitian products (3) and (4) under various coupling regimes. As it is obvious from Fig. 5, the K_2 entropy is always non-zero, which implies chaos. Furthermore, the strength of chaotic behavior intensely increases with the increase in the strength of interactions. Hence, going from the weak to the ultrastrong interaction regime, the system fully enters chaos. This strange phenomenon has not been observed elsewhere to date.

```
function k=K2d(X,d,ep,t)

k=1/t*log(Cde(X,d,ep)/Cde(X,d+1,ep));
```

(a)

```
function c=Cde(X,d,ep)

[M N]=size(X);

Cnt=0;
for n=1:N-d
    for m=n+1:N-d+1
        s=0;
        y=X(:,n:n+d-1)-X(:,m:m+d-1);
        for i=1:d
            s=s+norm(y(:,i))^2;
        end
        if sqrt(s)<ep
            Cnt=Cnt+1;
        end
    end
end

c=Cnt/N^2;
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(b)

Fig. 4. Subroutines used for calculation of K_2.

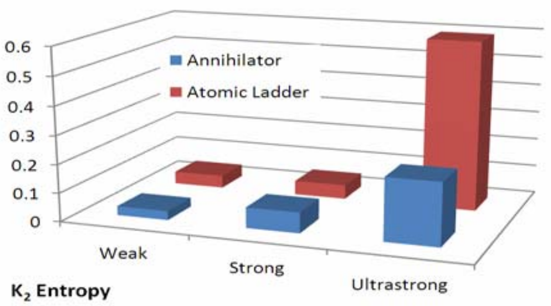

Fig. 5. Calculated Kolmogorov entropy K_2 for various interaction regimes [12].

IV. Conclusion

In this paper, we simulated a nine-partite cavity quantum electrodynamic system comprising of eight quantum dots and one cavity mode. Quantum dots, or field emitters, were allowed to undergo mutual dipole-dipole interactions with their nearest neighbors, and experience field-dipole interactions with the confined optical field. We calculated the Kolmogorov K_2 entropy for signals obtained from non-Hermitian products or expectation values and assessed the existence of chaos. We proved the emergence of quantum chaos under the ultrastrong coupling.

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