# Interaction of a Three-Level Atom ($\Lambda, V, \Xi$) with a Two-Mode Field Beyond Rotating Wave Approximation: Intermixed Intensity-Dependent Coupling

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**ABSTRACT- Recalling that the rotating wave approximation (RWA) is only valid in the weak coupling regimes, our purpose is to study the Hamiltonian dynamics of the interaction between various configurations of a three-level atom of $\Lambda -$, $V -$or $\Xi$-type distinctly with a two-mode radiation quantized field, while the RWA is not considered. Generally, this prevents one to achieve an analytical solution. Moreover, as we will show, using the perturbation theory analytical solution can be successfully obtained. According to our considerations, the contribution of counter rotating terms (CRTs) within the ordinary Hamiltonian is equivalent to arriving at some intermixed intensity-dependent atom-field coupling as functions of the two modes of the field, i.e., $f(\hat{n}_a,\hat{n}_b)$. At last, via evaluating the time-dependent atom–field state vector, the effects of CRTs on a few nonclassical properties of the state of the system as atomic population inversion and photon statistics are numerically studied. It is observed that, the presence of CRTs in the Hamiltonian dynamics destroys the clear patterns of collapse-revival phenomena in the time behavior of the evaluated quantities.**

**KEYWORDS:** Counter-rotating terms (CRTs), Intensity-dependent coupling, Nonclassical state, Perturbation theory, Rotating wave approximation (RWA).

## I. INTRODUCTION

The Jaynes–Cummings model (JCM) propounds a complete quantum approach for the description of the interaction of a two-level atom with a single-mode radiation field [1]. In recent decades, the atom-field interaction is widely studied, so that various generalizations of JCM have been introduced in the literature, including its extension to three-level atoms (like Λ , V, Ξ types [2]) interacting with quantized fields [3], various atom-field interacting systems with intensity-dependent couplings [4] and so on. It should be emphasized that in the JCM, rotating wave approximation (RWA) is taken into account; the fact that enables one to obtain an analytic solution for the Hamiltonian dynamics, while without it, usually one cannot solve the problem, analytically. To explain more, the RWA is a very popular technique in atomic physics, quantum optics and laser physics, since it leads to some simplifications in the calculation procedures. In this line, it is well-known that, the RWA is legitimatized to small ratios of the atom-field coupling divided by the atomic transition frequency [5]-[7]. Altogether, over the recent decades, new coupling regimes of the quantum Rabi model have been investigated, in which the coupling strengths of interaction can be enough strong [8], [9]. Such phenomena are typically observed experimentally in several systems [10], [11]. For instance, experimental progresses in the field reveal that for some of systems like trapped ions [12], cooper-pair boxes [13], [14] and flux qubits [15] enough strong coupling is accessible, and so the RWA clearly breaks down [16], [17].

Based on the above-mentioned observations, CRTs (virtual-photon processes) receive increasing attention of the authors in various systems, which yield substantial and interesting physical effects. Peng and Li have numerically analyzed the effect of the virtual-photon field on phase fluctuation and the periodic collapse revivals of atomic population in the JCM [18]-[20]. In [21] the entanglement dynamics of two atoms interacting with a dissipative coherent cavity field without RWA has been investigated. The entanglement evolution of two independent atoms without RWA has been proposed in [22] and the influence of CRTs in the JCM on the atomic population inversion has been investigated in [23]. It is shown that, the inclusion of CRTs to the Hamiltonian results in a transient entanglement between two atoms coupled to a standing-wave single-mode cavity field and the main source of entanglement is the two-photon coherence induced by two-photon transitions through virtual states [24].

Although considering CRTs causes serious problems in the analytical solution of the atom-field systems, these terms may be smoothly removed by perturbation theory approach [25], [26]. In this regard, recently in [27], [28], an analytical method, based on perturbation theory, has been presented to describe the interaction between a two-level atom and a single-mode field in the absence of RWA. It is shown that, the constant atom-field coupling and detuning parameters in the atom-field interaction without RWA have been converted to intensity-dependent terms, but fortunately without CRTs. In this respect, due to the existence of two different transitions in the three-level atoms ( $\Lambda, V, \Xi$ ), more diverse quantum phenomena may be expected from the such systems [29]. Therefore, more recently, using the same approach, one of us studied the interaction of a $\Lambda-$type atomwithout RWA, however still with a single-mode radiation field [30]. As a result, the two different coupling constants, namely $g_1, g_2$ have been changed respectively, to two specific intensity-dependent coupling functions $f_1(\hat{n})$, $f_2(\hat{n})$ which clearly depend on the number of photons. The authors also shown that the nonlinearity function $f_1(\hat{n})$ ( $f_2(\hat{n})$ ) not only depends on $g_1(g_2)$ but also on $g_2$ ($g_1$).

In this line, the interaction between a three-level atom with a single-mode field (in the presence of detuning) is not physically well understandable. Therefore, in this work we aim to develop and improve the work in [30] to two-mode quantized field, considering all well-known three-level atoms. In more detail, in this study we encouraged to investigate the interaction of three-level atoms ($\Lambda -$ , $V -$ , $\Xi -$ type) individually without RWA, while a two-mode quantized field is considered. Our calculations yield three different intensity-dependent functions for the three configurations of three-level atoms. As is seen, the two coupling constants $g_1, g_2$ have been changed respectively to two distinct intensity-dependent coupling functions $f_1$ **(** $\hat{n}_1$ **,** $\hat{n}_2$ **)**, $f_2$**(**$\hat{n}_1$**,**$\hat{n}_2$**)**which surprisingly depend on the number of photons of the two modes, this is while each mode of the field interacts with a particular transition in atoms. As is observed the nonlinearity function $f_1$**(**$\hat{n}_1$,$\hat{n}_2$**)** ($f_2$**(**$\hat{n}_1$,$\hat{n}_2$**)**) not only depends on $\hat{n}_1$, $g_1(n_2, g_2)$, but also on $\hat{n}_2$,$g_2$ ($\hat{n}_1$, $g_1$). In this way, for each type of three-level atoms we will arrive at two intertwined nonlinearity functions.

At this point it should be mentioned that atom-field intensity dependent function has a long background in the literature started by Buck and Sukumar [31], [32] and followed by others [33]-[36]. It is finally worth to mention that, our used approach simply transforms the unsolvable Hamiltonians to some solvable ones. Therefore, we can easily investigate the effect of CRTs on the physical properties of the obtained state vector of the system. To do this task, we will pay our attention to the atomic population inversion and the photon statistics of the field, which clarify the nonclassicality feature of the quantum state.

The paper is organized as follows. In Section 2 we solve analytically the dynamics of the interaction between all types of three-level atoms with a two-mode field without RWA, using the perturbation theory. In the next Section, we discuss the effect of CRTs on the dynamics of the atomic population inversion and the Mandel parameter. Finally, in Section 4 we present a summary and concluding remarks.

## II. The Interactions of Three-Level Atoms with Two-Mode Fields and Their Analytical Solutions

The purpose of this paper is to solve the dynamics of various configurations of three-level atoms ($\boldsymbol{\Lambda}$, $\boldsymbol{V}$, $\boldsymbol{\Xi}$) which distinctly interact with a two-mode quantized field, while the RWA is not considered (or equivalently, CRTs are taken into account). We suppose that the atom-field coupling is so strength that one cannot ignore the CRTs. As a result, at first glance it seems that the associated interaction Hamiltonians cannot be solved analytically. However, in the continuation, via following the perturbation theory approach, analytical solution for the Hamiltonian dynamics of each type of the atoms can be achieved.

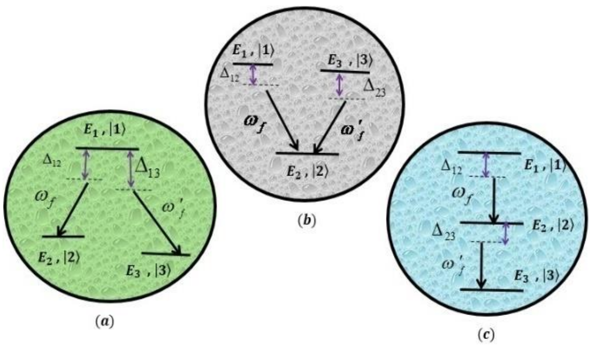


Fig. 1. Three-level atomic structure for (a) Λ-type, (b) $V$-type and (c) Ξ-type, with two-mode field.

### A. *Λ −Type Three-Level Atom without RWA*

We begin with the Λ-type three-level atom, where its atomic levels are labeled as the ground states $|2\rangle$, $|3\rangle$ and the excited state $|1\rangle$. The transitions between ground states $|2\rangle$ and state $|3\rangle$ are forbidden, while between $|1\rangle \leftrightarrow |3\rangle$ and $|1\rangle \leftrightarrow |2\rangle$ are allowed in the dipole approximation (see Fig. 1 (a)). In fact, we introduce a model in which a two-mode quantized radiation field interacts with a Λ-type atom. The Hamiltonian of the system (containing the CRTs) with electric dipole approximation reads as ($\hbar$=1):

$$\hat{H}=\hat{H}_0+\hat{H}_1+\hat{H}_2, \tag{1}$$

where

$$\hat{H}_0=\omega_f\hat{a}^\dagger\hat{a}+\omega_{f'}\hat{b}^\dagger\hat{b}+\sum_{j=1.2.3}E_j\,\hat{\sigma}_{jj}, \tag{2}$$

$$\hat{H}_1=g_1\left(\hat{a}^\dagger\hat{\sigma}_{21}+\hat{a}\hat{\sigma}_{12}\right)+g_2\left(\hat{b}^\dagger\hat{\sigma}_{31}+\hat{b}\hat{\sigma}_{13}\right), \tag{3}$$

$$\hat{H}_2=g_1\left(\hat{a}^\dagger\hat{\sigma}_{12}+\hat{a}\hat{\sigma}_{21}\right)+g_2\left(\hat{b}^\dagger\hat{\sigma}_{13}+\hat{b}\hat{\sigma}_{31}\right), \tag{4}$$

where $E_j$ is the energy of jth atomic level, $\hat{\sigma}_{ij}=ij(i,j$=1,2,3) denotes the atomic ladder operator between levels $i$ , $j$ , $\hat{a}$ and $\hat{a}^\dagger$ ( $\hat{b}$ , $\hat{b}^\dagger$ ) are respectively the bosonic annihilation and creation operators of the field, $\omega_f$ and $\omega_{f'}$ are the frequencies of the field modes and $g_1$, $g_2$ denote the coupling strengths between the field and the two allowed atomic transitions.

Usually in the literature, $\hat{H}_2$ which consists of the CRTs is neglected; this approximation is known as the RWA. There exist some reasons for the interests in this approximation, among them, lacking an analytical solution of the Hamiltonians containing the CRTs is remarkable. Moreover, in large coupling strengths, the CRTs are not negligible. Therefore, our purpose of the paper is to arrive an analytical solution to the problem while the CRTs are also maintained. To achieve the purpose, inspired from [27]-[29], we apply the following three unitary transformations on $\hat{H}$ in (1), successively. At first,

$$\hat{U}_1=e^{\varepsilon_1(\hat{a}^\dagger\hat{\sigma}_{12}-\hat{a}\hat{\sigma}_{21})+\varepsilon_2(\hat{b}^\dagger\hat{\sigma}_{13}-\hat{b}\hat{\sigma}_{31})}, \tag{5}$$

where

$$\varepsilon_1=\frac{g_1}{2\tilde{\omega}_{12}},\quad \tilde{\omega}_{12}=\frac{E_1-E_2+\omega_f}{2},\quad \varepsilon_2=\frac{g_2}{2\tilde{\omega}_{13}},\quad \tilde{\omega}_{13}=\frac{E_1-E_3+\omega_{f'}}{2}, \tag{6}$$

and then,

$$\hat{U}_2=e^{\beta_1\left(\hat{a}^{\dagger 2}-\hat{a}^2\right)\hat{\sigma}_z^{12}+\beta_2\left(\hat{b}^{\dagger 2}-\hat{b}^2\right)\hat{\sigma}_z^{13}}, \tag{7}$$

where

$$\beta_1=\frac{\varepsilon_1 g_1}{2\omega_f},\quad \beta_2=\frac{\varepsilon_2 g_2}{2\omega_{f'}}, \tag{8}$$

and finally,

$$\hat{U}_3=e^{\gamma_1\left(\hat{a}^{\dagger^3}\hat{\sigma}_{21}-\hat{a}^3\hat{\sigma}_{12}\right)+\gamma_2\left(\hat{b}^{\dagger^3}\hat{\sigma}_{31}-\hat{b}^3\hat{\sigma}_{13}\right)}, \qquad (9)$$

where

$$\gamma_1=\frac{2\varepsilon_1^2\tilde{\omega}_{12}+\varepsilon_2^2\tilde{\omega}_{13}}{\omega_f(E_1-E_2-3\omega_f)}g_1, \quad \gamma_2=\frac{2\varepsilon_2^2\tilde{\omega}_{13}+\varepsilon_1^2\tilde{\omega}_{12}}{\omega_{f'}(E_1-E_3-3\omega_{f'})}g_2, \qquad (10)$$

With the help of the mentioned transformations and through a lengthy calculations, we can arrive at the following effective Hamiltonian ( $\hat{n}_1=\hat{a}^\dagger\hat{a}$ and $\hat{n}_2=\hat{b}^\dagger\hat{b}$ are the number operators):

$$\begin{aligned}\hat{H}_{\text{eff}}&=\hat{U}_3^\dagger\hat{U}_2^\dagger\hat{U}_1^\dagger\hat{H}\hat{U}_1\hat{U}_2\hat{U}_3\approx\hat{H}_0+\hat{H}_1^D\\&+\varepsilon_1 g_1\left[\left(\hat{n}_1+\frac{1}{2}\right)\hat{\sigma}_z^{12}+\frac{\hat{\sigma}_{33}}{2}-\frac{1}{2}\right]\\&+\varepsilon_2 g_2\left[\left(\hat{n}_2+\frac{1}{2}\right)\hat{\sigma}_z^{13}+\frac{\hat{\sigma}_{22}}{2}-\frac{1}{2}\right],\end{aligned} \qquad (11)$$

where $\hat{H}_1^D$ in (11) is introduced as,

$$\begin{aligned}\hat{H}_1^D&=\hat{a}g_1f_1(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{12}+\hat{a}^\dagger\hat{\sigma}_{21}\\&+\hat{b}g_2f_2(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{13}+g_2f_2(\hat{n}_1,\hat{n}_2)\hat{b}^\dagger\hat{\sigma}_{31},\end{aligned} \qquad (12)$$

with the following definitions of the nonlinearity functions corresponding to intensity-dependent atom-field couplings,

$$\begin{aligned}f_1(\hat{n}_1,\hat{n}_2)&=\left(1-\varepsilon_1^2\hat{n}_1-\frac{\varepsilon_2^2}{2}\hat{n}_2\right),\\f_2(\hat{n}_1,\hat{n}_2)&=\left(1-\varepsilon_2^2\hat{n}_2-\frac{\varepsilon_1^2}{2}\hat{n}_1\right).\end{aligned} \qquad (13)$$

It may be emphasized that in our above calculations, higher orders of the perturbation parameters have been neglected according to the perturbation approach since we supposed that ($\varepsilon_1,\varepsilon_2 \ll 1$). Also, in applying the third transformation, it is assumed that we are far from three-photon transition; this means that $E_1-E_2=3\omega_f$, $E_1-E_3=3\omega_{f'}$. In this way, we arrived at an effective Hamiltonian that preserves the total excitation of quanta. This is the main advantage of the approach. By this we mean that, even though we encountered the CRTs in the Hamiltonian (see the initial Hamiltonian in Eqs. (1)-(4), moreover, an analytical solution can be reasonably expectable, just like the case in which we encountered the RWA in the atom-field systems. Also note that, in the effective Hamiltonian (11), the CRTs have been vanished and in the first-order (of $\varepsilon_1$ and $\varepsilon_2$), the third and fourth terms appear which describes the quantum shift in the energy of the atom–field system. In addition, it is noticeable that, in the second-order Hamiltonian (of $\varepsilon_1$ and $\varepsilon_2$), the considered coupling constants $g_1$ and $g_2$ have been converted to $g_1f_1(\hat{n}_1,\hat{n}_2)$ and $g_2f_2(\hat{n}_1,\hat{n}_2)$, respectively. Therefore, briefly, we showed that considering CRTs (i.e., JCM beyond RWA) our used approach straightforwardly transforms the unsolvable linear (constant atom-field coupling) interaction dynamics to a solvable nonlinear (intensity-dependent) coupling JCM.

### B. *$V-$ and $\Xi-$Type Three-Level Atoms with Two-Mode Field without RWA*

After presenting the detailed procedure of the interaction between a Λ-type atom with a two-mode field, containing the CRTs, in the previous subsection, in this subsection we want to consider the other two types of atoms, shortly. At first, we pay our attention to the $V$-type three-level atom where its two upper levels is allowed to transit to the lower level, and the transition between the two upper levels is forbidden (see Fig. 1(b)). By this description, the full Hamiltonian of the interacting system (containing the CRTs) in the electric dipoleapproximation reads as ($\hbar$= 1):

$$\hat{H}=\hat{H}_0+\hat{H}_1+\hat{H}_2, \qquad (14)$$

where $\hat{H}_0$ has been introduced in Eq. (2) and

$$\hat{H}_1=g_1\left(\hat{a}^\dagger\hat{\sigma}_{21}+\hat{a}\hat{\sigma}_{12}\right)+g_2\left(\hat{b}^\dagger\hat{\sigma}_{23}+\hat{b}\hat{\sigma}_{32}\right), \qquad (15)$$

$$\hat{H}_2=g_1\left(\hat{a}^\dagger\hat{\sigma}_{12}+\hat{a}\hat{\sigma}_{21}\right)+g_2\left(\hat{b}^\dagger\hat{\sigma}_{32}+\hat{b}\hat{\sigma}_{23}\right), \qquad (16)$$

As in the Λ-type atom, however via applying the following three successive unitary transformations,

$$\hat{U}_1=e^{\varepsilon_1(\hat{a}^\dagger\hat{\sigma}_{12}-\hat{a}\hat{\sigma}_{21})+\varepsilon_2(\hat{b}^\dagger\hat{\sigma}_{32}-\hat{b}\hat{\sigma}_{23})}, \qquad (17)$$

$$\hat{U}_2=e^{\beta_1(\hat{a}^{\dagger^2}-\hat{a}^2)\hat{\sigma}_z^{12}+\beta_2(\hat{b}^{\dagger^2}-\hat{b}^2)\hat{\sigma}_z^{23}}, \qquad (18)$$

$$\hat{U}_3=e^{\gamma_1\left(\hat{a}^{\dagger^3}\hat{\sigma}_{21}-\hat{a}^3\hat{\sigma}_{12}\right)+\gamma_2\left(\hat{b}^{\dagger^3}\hat{\sigma}_{23}-\hat{b}^3\hat{\sigma}_{32}\right)}, \qquad (19)$$

on the Hamiltonian (14), the following effective Hamiltonian may be obtained,

$$\hat{H}_{eff} \approx \hat{H}_0 + \hat{H}_1^D + \varepsilon_1 g_1 \left[\left(\hat{n}_1 + \frac{1}{2}\right)\hat{\sigma}_z^{12} + \frac{\hat{\sigma}_{33}}{2} - \frac{1}{2}\right] + \varepsilon_2 g_2 \left[\left(\hat{n}_2 + \frac{1}{2}\right)\hat{\sigma}_z^{32} + \frac{\hat{\sigma}_{22}}{2} - \frac{1}{2}\right], \quad (20)$$

with

$$\hat{H}_1^D = \hat{a} g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{12} + g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{a}^\dagger\hat{\sigma}_{21} + \hat{b} g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{32} + g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{b}^\dagger\hat{\sigma}_{23}, \quad (21)$$

where the nonlinearity functions which appear in the field variables describe the fact that, the constant couplings are transformed to intensity-dependent couplings as below,

$$g_1 \rightarrow g_1 f_1(\hat{n}_1.\hat{n}_2) = g_1\left[1 - \varepsilon_1^2\hat{n}_1 - \frac{\varepsilon_2^2}{2}(1+\hat{n}_2)\right],$$
$$g_2 \rightarrow g_2 f_2(\hat{n}_1.\hat{n}_2) = g_2\left[1 - \varepsilon_2^2\hat{n}_2 - \frac{\varepsilon_1^2}{2}(1+\hat{n}_1)\right], \quad (22)$$

Also note that the parameters $\varepsilon_1 = \frac{g_1}{2\widetilde{\omega}_{12}}, \left(\widetilde{\omega}_{12} = \frac{E_1 - E_2 + \omega_f}{2}\right)$ and $\varepsilon_2 = \frac{g_2}{2\widetilde{\omega}_{23}}, \left(\widetilde{\omega}_{13} = \frac{E_2 - E_3 + \omega_{f'}}{2}\right), \gamma_1 = \frac{2\varepsilon_1^2\widetilde{\omega}_{12} + \varepsilon_2^2\widetilde{\omega}_{23}}{\omega_f(E_1 - E_2 - 3\omega_f)} g_1, \gamma_2 = \frac{2\varepsilon_2^2\widetilde{\omega}_{23} + \varepsilon_1^2\widetilde{\omega}_{12}}{\omega_{f'}(E_2 - E_3 - 3\omega_{f'})} g_2$, and the parameters $\beta_1, \beta_2$ have been defined respectively in (8).

Now we want to repeat the procedure for a typical three-level atom in ladder (cascade) configuration as shown in Fig. 1(c). The permissible transitions are from the upper level $|1\rangle$ to the intermediate level $|2\rangle$ as well as from level $|2\rangle$ to the ground level $|3\rangle$. The Hamiltonian describing the interaction of ladder type atom with a two-mode field can be written as below, wherein we have kept the CRTs, too,

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2, \quad (23)$$

where $\hat{H}_0$ has been introduced in Eq. (2) and

$$\hat{H}_1 = g_1\left(\hat{a}^\dagger\hat{\sigma}_{12} + \hat{a}\hat{\sigma}_{21}\right) + g_2\left(\hat{b}^\dagger\hat{\sigma}_{32} + \hat{\sigma}_{23}\right), \quad (24)$$

$$\hat{H}_2 = g_1\left(\hat{a}^\dagger\hat{\sigma}_{21} + \hat{a}\hat{\sigma}_{12}\right) + g_2\left(\hat{b}^\dagger\hat{\sigma}_{23} + \hat{b}\hat{\sigma}_{32}\right), \quad (25)$$

In this case also we apply three successive unitary transformations defined as below,

$$\hat{U}_1 = e^{\varepsilon_1(\hat{a}^\dagger\hat{\sigma}_{12} - \hat{a}\hat{\sigma}_{21}) + \varepsilon_2(\hat{b}^\dagger\hat{\sigma}_{23} - \hat{b}\hat{\sigma}_{32})}, \quad (26)$$

$$\hat{U}_2 = e^{\beta_1\left(\hat{a}^{\dagger 2} - \hat{a}^2\right)\hat{\sigma}_z^{12} + \beta_2\left(\hat{b}^{\dagger 2} - \hat{b}^2\right)\hat{\sigma}_z^{13}}, \quad (27)$$

$$\hat{U}_3 = e^{\gamma_1\left(\hat{a}^{\dagger 3}\hat{\sigma}_{12} - \hat{a}^3\hat{\sigma}_{21}\right) + \gamma_2\left(\hat{b}^{\dagger 3}\hat{\sigma}_{23} - \hat{b}^3\hat{\sigma}_{32}\right)}, \quad (28)$$

which arrives one at the nonlinear (intensity-dependent) atom-field couplings in the dynamical Hamiltonian as below:

$$\hat{H}_{eff} \approx \hat{H}_0 + \hat{H}_1^D + \varepsilon_1 g_1 \left[\left(\hat{n}_1 + \frac{1}{2}\right)\hat{\sigma}_z^{12} + \frac{\hat{\sigma}_{33}}{2} - \frac{1}{2}\right] + \varepsilon_2 g_2 \left[\left(\hat{n}_2 + \frac{1}{2}\right)\hat{\sigma}_z^{23} + \frac{\hat{\sigma}_{11}}{2} - \frac{1}{2}\right], \quad (29)$$

with the definition,

$$\hat{H}_1^D = \hat{a} g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{12} + g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{a}^\dagger\hat{\sigma}_{21} + \hat{b} g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{23} + g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{b}^\dagger\hat{\sigma}_{32}, \quad (30)$$

where the appeared nonlinearity functions are such that, in this Eq. the constant couplings in the initial Hamiltonian (3), (4) have been replaced by some intensity-dependent coupling functions as is explained in below,

$$g_1 \rightarrow g_1 f_1(\hat{n}_1,\hat{n}_2) = g_1\left[1 - \varepsilon_1^2\hat{n}_1 - \frac{\varepsilon_2^2}{2}\hat{n}_2\right],$$
$$g_2 \rightarrow g_2 f_2(\hat{n}_1,\hat{n}_2) = g_2\left[1 - \frac{\varepsilon_1^2}{2}(1+\hat{n}_1) - \varepsilon_2^2\hat{n}_2\right], \quad (31)$$

Note that the parameters $\varepsilon_1(\varepsilon_2), \beta_1(\beta_2)$ and $\gamma_1(\gamma_2)$ have been defined in description of $V$ atom.

Note that in this subsection we only presented our calculations that led us to finding the nonlinearity functions associated with the $V-$ and $\Xi-$ type atoms. Our further calculations for arriving at the state vector of the system and also evaluating the atomic population inversion and photon statistics may be seen in the Appendix of the paper.

### C. *The Eigenvalues of the Hamiltonians Based on Perturbation Theory*

In this section, via working with the final form of the obtained Hamiltonians in the previous section, the eigenvalues and eigenstates of the atom-field system will be obtained. We begin with the Λ-type atom, meanwhile, instead of addressing the Hamiltonian (11), it is more favorable to rewrite it as follows:

$$\begin{aligned}\hat{H}_{eff}=&\omega_f\hat{n}_1+\omega_{f'}\hat{n}_2+\tfrac{1}{3}\omega_f\hat{\sigma}_z^{12}+\tfrac{1}{3}\omega_{f'}\hat{\sigma}_z^{13}\\&+\ \frac{(2\Delta_{12}-\Delta_{13})}{3}\hat{\sigma}_z^{12}+\frac{(2\Delta_{13}-\Delta_{12})}{3}\hat{\sigma}_z^{13}\\&+\tfrac{1}{3}(E_1+E_2+E_3)(\hat{\sigma}_{11}+\hat{\sigma}_{22}+\hat{\sigma}_{33})\\&+\varepsilon_1 g_1(\hat{n}_1\hat{\sigma}_z^{12}-\hat{\sigma}_{22})+\varepsilon_2 g_2(\hat{n}_2\hat{\sigma}_z^{13}-\hat{\sigma}_{33})\\&+\hat{a}g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{12}+g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{a}^\dagger\hat{\sigma}_{12}\\&+\hat{b}g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{13}+g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{b}^\dagger\hat{\sigma}_{31},\end{aligned} \quad (32)$$

where the detuning parameters are defined as,

$$\Delta_{12}=E_1-E_2-\omega_f, \qquad \Delta_{13}=E_1-E_3-\omega_{f'} \quad (33)$$

and with the help of the following relations:

$$\begin{aligned}&\hat{N}=\omega_f\hat{n}_1+\omega_{f'}\hat{n}_2+\tfrac{1}{3}\omega_f\hat{\sigma}_z^{12}+\tfrac{1}{3}\omega_{f'}\hat{\sigma}_z^{13},\\&\hat{\sigma}_z^{12}=\hat{\sigma}_{11}-\hat{\sigma}_{22}, \qquad \hat{\sigma}_z^{13}=\hat{\sigma}_{11}-\hat{\sigma}_{33}.\end{aligned} \quad (34)$$

Based on the fact that $[\hat{N},\hat{H}_{eff}]=0$, which leads one to the realization that $\hat{N}$ is a constant of motion, we can straightforwardly solve the eigenvalue equation associated with the Hamiltonian (29) for a constant quanta $N$. We consider the eigenvalue equation of effective Hamiltonian as follows:

$$\hat{H}_{eff}|\phi_L\rangle_{eff}=\mu_L|\phi_L\rangle_{eff}, \qquad L=1,2,3, \quad (35)$$

Hence, for a fixed number of quanta $N=n+1$, and in the unpaired atom-field states, $|1,n_1,n_2\rangle\equiv|1\rangle\otimes|n_1,n_2\rangle$, $|2,n_1+1,n_2\rangle\equiv|2\rangle\otimes|n_1+1,n_2\rangle$, $|3,n_1,n_2+1\rangle\equiv|3\rangle\otimes|n_1,n_2+1\rangle$, the eigenvalues can be estimated from the following matrix relation:

$$\begin{vmatrix}h_1+h_2+D_1+D_2-\mu_L & \sqrt{n_1+1}f_1(n_1+1,n_2) & \sqrt{n_2+1}f_2(n_1,n_2+1)\\ \sqrt{n_1+1}f_1(n_1+1,n_2) & h_2-D_1-\mu_L & 0\\ \sqrt{n_2+1}f_2(n_1,n_2+1) & 0 & h_3-D_2-\mu_L\end{vmatrix}=0, \quad (36)$$

Here we have defined:

$$\begin{aligned}&h_1=\omega_f\left(n_1+\tfrac{1}{3}\right)+\omega_{f'}\left(n_2+\tfrac{1}{3}\right)g_1\hbar-\varepsilon_2 g_2,\\&h_2=\omega_f\left(n_1+\tfrac{1}{3}\right)+\omega_{f'}\left(n_2+\tfrac{1}{3}\right)-\varepsilon_1 g_1\hbar+\frac{(\Delta_{13}-\Delta_{12})}{3},\\&h_3=\omega_f\left(n_1+\tfrac{1}{3}\right)+\omega_{f'}\left(n_2+\tfrac{1}{3}\right)-\varepsilon_2 g_2\hbar+\frac{(\Delta_{12}-\Delta_{13})}{3},\\&D_1=\frac{\Delta_{12}}{3}+\varepsilon_1 g_1(n_1+1),\\&D_2=\frac{\Delta_{13}}{3}+\varepsilon_2 g_1(n_2+1).\end{aligned} \quad (37)$$

Using some algebraic calculations, the above equation can be reduced to the following cubic equation,

$$\mu_L^3+x_1\mu_L^2+x_2\mu_L+x_3=0, \qquad L=1,2,3, \quad (38)$$

where $\mu_L$ is the eigenvalue that corresponds to $L$th eigenvector of $\hat{H}_{eff}$ and

$$\begin{aligned}x_1=&-(h_1+h_2+h_3),\\x_2=&-(n_1+1)f_1^2(n_1+1,n_2)\\&-(n_2+1)f_2^2(n_1,n_2+1)+h_1h_2\\&+h_1h_3+h_2h_3+(h_2-h_1)D_1\\&+(h_3-h_1)D_2-D_1D_2-D_1^2-D_2^2,\\x_3=&(n_1+1)f_1^2(n_1+1,n_2)h_3\\&+(n_2+1)f_2^2(n_1,n_2+1)h_2\\&-h_1h_2h_3+h_1h_3D_1-h_2h_3D_1\\&+h_1h_2D_2-h_2h_3D_2+h_3D_1^2\\&+h_2D_2^2-D_2D_1^2-D_1D_2^2\\&+(h_2+h_3-h_1)D_1D_2\\&-(n_1+1)f_1^2(n_1+1,n_2)D_2\\&-(n_2+1)f_2^2(n_1,n_2+1).\end{aligned} \quad (39)$$

To solve the cubic algebraic equation, we used the Cardano formula [37], according to which the eigenvalues of $\hat{H}_{eff}$ are given by:

$$\mu_L(n)=-\tfrac{1}{3}x_1+\tfrac{2}{3}\sqrt{x_1^2-3x_2}\,\cos\left[\theta(n)+\tfrac{2}{3}(L-1)\pi\right], \quad (40)$$

$$\theta(n)=\tfrac{1}{3}\cos^{-1}\left(\frac{9x_1x_2-2x_1^3-27x_3}{2(x_1^2-3x_2)^{\frac{3}{2}}}\right). \quad (41)$$

Also for the corresponding eigenvectors one finds:

$$\left|\phi_l\right\rangle_{eff}=a_l(n,m)|1,n_1,n_2\rangle+b_l(n,m)|2,n_1+1,n_2\rangle, \\ +c_l(n,m)|3,n_1,n_2+1\rangle, \quad (42)$$

Hence the eigenvalue equation (35) can be written in a matrix form as below:

$$\begin{pmatrix} h_1+h_2+D_1+D_2-\mu_L & \sqrt{n_1+1}f_1(n_1+1,n_2) & \sqrt{n_2+1}f_2(n_1,n_2+1) \\ \sqrt{n_1+1}f_1(n_1+1,n_2) & h_2-D_1-\mu_L & 0 \\ \sqrt{n_2+1}f_2(n_1,n_2+1) & 0 & h_3-D_2-\mu_L \end{pmatrix}$$

$$\begin{pmatrix} a_l \\ b_l \\ c_l \end{pmatrix}=\mu_L\begin{pmatrix} a_l \\ b_l \\ c_l \end{pmatrix}, \quad (43)$$

By combining the above equations and the probability conservation relation:

$$|a_l(n_1,n_2)|^2+|b_l(n_1,n_2)|^2+|c_l(n_1,n_2)|^2=1, \quad (44)$$

the coefficients in (44) are obtained as follows:

$$a_l(n_1,n_2)= \left(1+\frac{(n_1+1)f_1^2(n_1+1,n_2)}{(\mu_L(n_1,n_2)-h_2+D_1)^2}+\frac{(n_2+1)f_2^2(n_1,n_2+1)}{(\mu_L(n_1,n_2)-h_3+D_2)^2}\right)^{-\frac{1}{2}},$$

$$b_l(n_1,n_2)=\frac{\sqrt{n_1+1}f_1(n_1+1,n_2)}{\mu_L(n_1,n_2)-h_2+D_1}a_l(n_1,n_2),$$

$$c_l(n_1.n_2)=\frac{\sqrt{n_1+1}f_2(n_1,n_2+1)}{\mu_L(n_1.n_2)-h_3+D_2}a_l(n_1,n_2), \quad (45)$$

Now we consider the eigenvalue equation for $\widehat{H}$, which is defined by Eq. (1) in the form:

$$\widehat{H}\left|\phi_L\right\rangle_d=\mu_L\left|\phi_L\right\rangle_d, \qquad L=1,2,3, \quad (46)$$

Keeping in mind the fundamentals of perturbation theory, it seems reasonable that the eigenvalues $\mu_l$ are approximated to the eigenvalues $\mu_L$. Also, according to the transformations which are applied in the previous section, the following relationship is established between the eigenvectors of $\widehat{H}$ and $\widehat{H}_{eff}$:

$$\left|\phi_L\right\rangle_d=U_1^\dagger U_2^\dagger U_3^\dagger\left|\phi_L\right\rangle_{eff}\approx a_l(n_1,n_2)|1,n_1,n_2\rangle \\ +b_l(n_1,n_2)|2,n_1+1,n_2\rangle \\ +c_l(n_1,n_2)|3,n_1,n_2+1\rangle \\ -\varepsilon_1\sqrt{n_1}a_l(n_1,n_2)|2,n_1-1,n_2\rangle \\ -\sqrt{n_1+2}b_l(n_1,n_2)|1,n_1+2,n_2+1\rangle \\ -\varepsilon_2\sqrt{n_2}a_l(n_1,n_2)|3,n_1-1,n_2\rangle \\ \sqrt{n_2+2}c_l(n_1,n_2)|1,n_1+1,n_2+2\rangle, \quad (47)$$

## D. *The State Vector of the Entire System*

In this subsection, we first assume that the atom is prepared in the state basis $|1\rangle$ at the onset of interaction and the field is a bimodal coherent state. So, the initial state of the atom-field system reads as,

$$|\psi(0)\rangle=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1}P_{n_2}|1,n_1,n_2\rangle, \\ P_{n_i}=\frac{e^{\frac{-|\alpha|^2}{2}}\alpha^{n_i}}{\sqrt{n_i!}}, \quad i=n_1,n_2 \quad (48)$$

where $|\alpha|^2$ is the average number of photons in the field at $t=0$. One can easily obtain the atom-field state vector of the system at time $t$ by applying the time evolution operator onto the assumed initial state:

$$|\psi(t)\rangle=e^{-\frac{i}{\hbar}\widehat{H}t}|\psi(0)\rangle= \\ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{L=1}^{3}e^{-\frac{i\mu_L(n_1,n_2)t}{\hbar}}\lambda_l(n_1,n_2)\left|\phi_l\right\rangle, \quad (49)$$

$$\lambda_l(n_1,n_2)=\left\langle\phi_L\middle|\psi(0)\right\rangle=a_l(n_1,n_2)P_{n_1}P_{n_2} \\ -\varepsilon_1 b_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2+1} \\ -\varepsilon_2 c_l(n_1,n_2)\sqrt{n_2+2}P_{n_1+1}P_{n_2+2}, \quad (50)$$

the state vector at any time, up to the first-order approximation may be obtained in the following form:

$$|\psi(t)\rangle=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(A_{n_1,n_2}(t)|1,n_1,n_2\rangle \\ +B_{n_1+1,n_2}(t)|2,n_1+1,n_2\rangle \\ +C_{n_1,n_2+1}(t)|3,n_1,n_2+1\rangle, \quad (51)$$

where the probability amplitudes $A_{n_1,n_2}(t)$, $B_{n_1+1,n_2}(t)$ and $C_{n_1,n_2+1}(t)$ in Eq. (51) have been determined by the following relations:

$$A_{n_1,n_2}(t)=\sum_{l=1}^{3}e^{-i\mu_L(n_1,n_2)t}(a_l(n_1,n_2)^2P_{n_1}P_{n_2} \\ -\varepsilon_1 b_l(n_1,n_2)a_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2} \\ -\varepsilon_2 a_l(n_1,n_2)c_l(n_1,n_2)\sqrt{n_2+2}P_{n_1}P_{n_2+2}) \\ -e^{-i\mu_L(n_1-2,n_2)t}\varepsilon_1 b_l(n_1-2,n_2)a_l(n_1-2,n_2)\sqrt{n_1}P_{n_1-2}P_{n_2} \\ +e^{-i\mu_L(n_1,n_2-2)t}P_{n_1}P_{n_2-2}, \quad (52)$$

$$B_{n_1+1,n_2}(t)= \sum_{l=1}^{3} e^{-i\mu_L(n_1,n_2)t}(a_l(n_1,n_2)b_l(n_1,n_2)P_{n_1}P_{n_2} -\varepsilon_1 b_l(n_1,n_2)^2\sqrt{n_1+2}P_{n_1+2}P_{n_2} -\varepsilon_2 b_l(n_1,n_2)c_l(n_1,n_2)\sqrt{n_2+2}P_{n_1}P_{n_2+2} +e^{-i\mu_L(n_1+2,n_2)t}\varepsilon_1 a_l(n_1-2,n_2)^2\sqrt{n_1+2}P_{n_1+2}P_{n_2} -e^{-i\mu_L(n_1,n_2+2)t}\varepsilon_2\sqrt{n_2+2}a_l(n_1,n_2-2)^2P_{n_1}P_{n_2+2}, \quad (53)$$

$$C_{n_1,n_2+1}(t) = \sum_{l=1}^{3} e^{-i\mu_L(n_1,n_2)t}\left(a_l(n_1,n_2)c_l(n_1,n_2)P_{n_1}P_{n_2} -\varepsilon_1 b_l(n_1,n_2)c_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2} -\varepsilon_2 c_l(n_1,n_2)^2\sqrt{n_2+2}P_{n_1}P_{n_2+2}\right) +e^{-i\mu_L(n_1+2,n_2)t}\varepsilon_1 a_l(n_1-2,n_2)^2\sqrt{n_1+2}P_{n_1+2}P_{n_2} -e^{-i\mu_L(n_1,n_2+2)t}\varepsilon_2\sqrt{n_2+2}a_l(n_1,n_2-2)^2P_{n_1}P_{n_2+2}, \quad (54)$$

It should be noted that due the used approximations in the solving processes, normalization of the state vector should be corrected. In order to be certain about this fact, we rewrite the system state as:

$$|\psi(t)\rangle_{norm} = N(t)|\psi(t)\rangle, \quad (55)$$

where

$$N(t)=\langle\psi(t)|\psi(t)\rangle^{-\frac{1}{2}} = \begin{pmatrix} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(|A_{n_1,n_2}(t)|^2+ \\ |B_{n_1+1,n_2}(t)|^2+|C_{n_1,n_2+1}(t)|^2 \end{pmatrix}^{-\frac{1}{2}}. \quad (56)$$

## III. Nonclassical Properties of the System and the Effects of CRTs

Searching for nonclassical properties of the atom-field interactions is of enough interest [38]-[40]. In this section we use a few nonclassical features including the population inversion of atomic levels and photon statistics of the quantized field, with and without RWA. For this purpose, we plot the latter quantities in terms of rescaled time for constant amounts of $\frac{\Delta_{12}}{\omega_f} = 0.2, \frac{\Delta_{13}}{\omega_{f'}} = 0.28$ and $|\alpha|^2 = 25$. Then, we compare the results from the plotted figures to investigate the effect of the CRTs on the time evolution of the dynamical properties of the state of the system.

### A. *Population Inversion of Atoms*

Atomic population inversion concerning the considered atomic systems indicates difference between higher level population(s) with the total population of the lower level(s), i.e.,

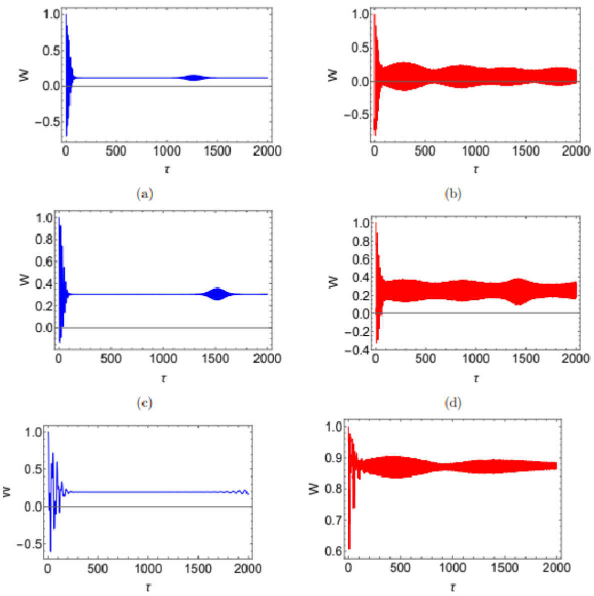


Fig. 2. The time evolution of the atomic population inversion versus the scaled time $\tau = \omega_f t$. Here, we considered $\frac{\Delta_{12}}{\omega_f} = 0.2, \frac{\Delta_{13}}{\omega_{f'}} = 0.28$, $|\alpha|^2 = 25$ and $\frac{g_1}{\omega_f} = 0.01$, $\frac{g_2}{\omega_{f'}} = 0.04$. The left curves correspond to the presence of RWA ($\varepsilon_1 = \varepsilon_2 = 0$) and the right curves are obtained in the absence of RWA ($\varepsilon_1 = \frac{1}{55}, \varepsilon_2 = \frac{1}{57}$). Top plots are related to the $\Lambda$, middled plots for the $V$ and low plots are for the ladder atom.

$$W(t)=N(t)^2 \left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(|A_{n_1,n_2}(t)|^2 -|B_{n_1+1,n_2}(t)|^2-|C_{n_1,n_2+1}(t)|^2\right). \quad (57)$$

In Fig. 2 we plotted atomic inversion for the three configurations of atoms with RWA (left plots) and without RWA(right plots) in terms of the scaled time $\tau = \omega_f t$ for particular chosen parameters. According to our numerical results, it is observed that in the presence RWA (left plots) the nonoscillative collapse regions are replaced by rapidly oscillations with finite amplitudes in the absence of RWA (right plots). In fact, these rapidly oscillations are caused by the effect of entering the intensity-dependent coupling, nonlinearity regimes, virtualphoton processes. Accordingly, considering the CRTs, the clear patterns of collapse-revivals with RWA are drastically destroyed.

### B. *Photon Statistics of the Fields: Mandel Parameter*

In order to examine the photon statistics of the field during the interaction with each of the three-level atoms, the Mandel parameter which is defined as follows:

$$Q=\frac{\langle \hat{n}^2\rangle-\langle \hat{n}\rangle^2}{\langle \hat{n}\rangle}-1, \quad (58)$$

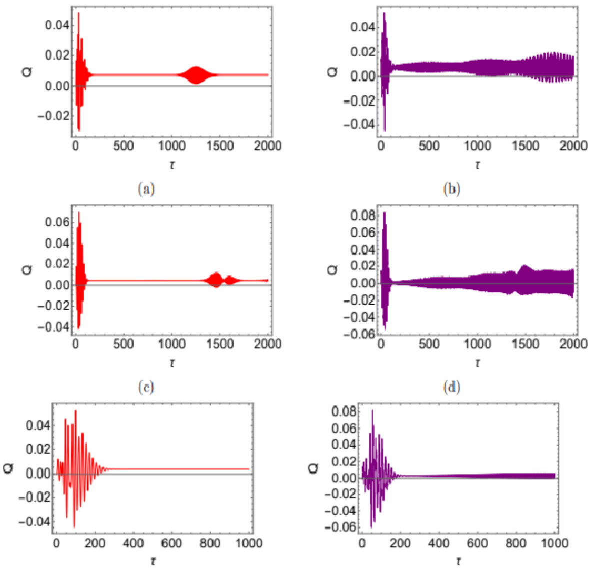

Fig. 3. The time evolution of Mandel parameter for chosen parameters similar to Fig. 2.

If for a field state, $-1 \leq Q < 0$, the photon statistics are sub-Poissonian (non-classical light), if $Q = 0$, the field possesses Poissonian distribution (standard coherent light) and if $Q > 0$ the field has super-Poissonian distribution (classical light). The temporal evolution of the Mandel parameter in terms of the scaled time is shown in Fig. 3. From the numerical results one finds that, the relaxation regions (collapses) in case with RWA (left plots), have been replaced by the rapidly fluctuations (partial revivals) resulting from the virtual-photon processes of CRTs (right plots) as well as the presence of nonlinearity functions in the atom-field coupling. Briefly, what is clearly observed is that, similar to the case of atomic population inversion, considering the CRTs clearly changes the pattern of collapse-revivals phenomena. Also comparing the three configurations of atoms, it is observed that the effects of CRTs is more visible in the $\Lambda-$, $V-$type atoms relative to $\Xi$-type, either in Fig. 2 or Fig. 3.

## IV. Summary and Conclusion

In this paper, based on the perturbation theory approach, we considered the quantum interaction between a three-level atom with a two-mode quantized field without considering RWA. We solved analytically the Hamiltonian interaction corresponding to all types of three-level atoms ($\Lambda.V.\Xi$). At first, as is shown, the contribution of CRTs in the first-order approximation appears as intensity-dependent displacement in the energy of the atom-field levels and in the second-order two intensity-dependent functions appear instead of the coupling constants. By solving the effective Hamiltonian dynamics, we obtained the associated eigenvalues and the eigenstates, and through them, we arrived at the approximate eigenvalues and the eigenstates Hamiltonian of the interaction system. In the continuation of the paper, we investigated the temporal evolution of the two physical quantities associated with the considered systems in the presence and absence of RWA based on the obtained state vector of the atom-field systems. As is observed via the numerical results, the population inversion and Mandel parameter both have collapse and revival patterns. However, while in the presence of RWA (left plots) we observed clear collapse-revival patterns, in the absence of RWA (right plots) one arrived at typical patterns of collapse-revival. In fact the presence of CRTs destroys the apparent patterns of collapse-revivals. In general, the observed effects can be attributed to virtual (photons) transitions created by CRTs in the atom-field system Hamiltonian. Altogether, about the nonclassicality signs we observed that in both cases, either with or without CRTs and in all configurations of atoms, the considered systems possess such nonclassicalities clearly. These results are expectable and consistence with the previous observations in this subject [27], [30].

## APPENDIX

In this appendix, the calculations of the eigenvalues and eigenstates of the atom-field system related to ladder and V atoms are presented. In the following, by obtaining the state vector of the entire system, the calculations related to the non-classical properties of these two types of atoms are also are presented.

First the ladder atoms

$$\begin{aligned}\hat{H}_{eff}=&\omega_f\hat{n}_1+\omega_{f'}\hat{n}_2+\omega_f\hat{\sigma}_z^{12}+\omega_{f'}\hat{\sigma}_z^{23}\\&+\tfrac{(2\Delta_{12}+\Delta_{23})}{3}\hat{\sigma}_z^{12}+\tfrac{(2\Delta_{23}+\Delta_{12})}{3}\hat{\sigma}_z^{23}\\&+\tfrac{1}{3}(E_1+E_2+E_3)(\hat{\sigma}_{11}+\hat{\sigma}_{22}+\hat{\sigma}_{33})\\&+\varepsilon_1 g_1\left(\hat{n}_1\hat{\sigma}_z^{12}-\hat{\sigma}_{22}\right)+\varepsilon_2 g_2\\&+\hat{a}g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{12}+g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{a}^\dagger\hat{\sigma}_{21}\\&+\hat{b}g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{23}+g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{b}^\dagger\hat{\sigma}_{32},\end{aligned}\quad (1a)$$

$$\begin{aligned}&\Delta_{12}=E_1-E_2-\omega_f,\qquad \Delta_{23}=E_2-E_3-\omega_{f'},\\&\hat{N}'=\omega_f\hat{n}_1+\omega_{f'}\hat{n}_2+\omega_f\hat{\sigma}_z^{12}+\omega_{f'}\hat{\sigma}_z^{23},\\&\hat{\sigma}_z^{12}=\hat{\sigma}_{11}-\hat{\sigma}_{22},\hat{\sigma}_z^{23}=\hat{\sigma}_{22}-\hat{\sigma}_{33},\end{aligned}\quad (2a)$$

Hence, for a fixed quantum number $N=n+1$, and in unpaired atom-field states, $|1,n_1,n_2\rangle\equiv|1\rangle\otimes$, $|2,n_1+1,n_2\rangle\equiv|2\rangle\otimes|n_1+1,n_2\rangle$, $|3,n_1,n_2+1\rangle\equiv|3\rangle\otimes|n_1+1,n_2+1\rangle$, the eigenvalues can be estimated from the following matrix relation:

$$\begin{vmatrix}h_1-\mu_L & \sqrt{n_1+1}f_1(n_1+1,n_2) & 0\\ \sqrt{n_1+1}f_1(n_1+1,n_2) & h_2-\mu_L & \sqrt{n_2+1}f_2(n_1,n_2+1)\\ 0 & \sqrt{n_2+1}f_2(n_1,n_2+1) & h_3-\mu_L\end{vmatrix}=0,\quad (3a)$$

where we have defined:

$$h_1=\omega_f\left(n_1+\tfrac{1}{2}\right)+\omega_{f'}\left(n_2+\tfrac{1}{2}\right)+\varepsilon_1 g_1 n_1+\tfrac{(2\Delta_{12}+\Delta_{23})}{3},$$

$$\begin{aligned}h_2=&\omega_f\left(n_1+\tfrac{1}{2}\right)+\omega_{f'}\left(n_2+\tfrac{1}{2}\right)-\varepsilon_1 g_1\\&-\varepsilon_1 g_1 n_1+\varepsilon_2 g_2 n_2+\tfrac{(\Delta_{13}-\Delta_{12})}{3},\end{aligned}$$

$$\begin{aligned}h_3=&\omega_f\left(n_1+\tfrac{1}{2}\right)+\omega_{f'}\left(n_2+\tfrac{1}{2}\right)\\&-\varepsilon_2 g_2-\varepsilon_2 g_2 n_2-\tfrac{(\Delta_{12}+2\Delta_{23})}{3}.\end{aligned}\quad (4a)$$

Using some algebraic calculations, the above equation is reduced to the following algebraic cubic equation:

$$\mu_L^3+x_1\mu_L^2+x_2\mu_L+x_3,\quad L=1,2,3\quad (5a)$$

where $\mu_L$ is the eigenvalue corresponding to the $Lth$ eigenvector of $\hat{H}_{eff}$ and

$$x_1=-(h_1+h_2+h_3),\quad (6a)$$

$$\begin{aligned}x_2=&-(n_1+1)f_1^2(n_1+1,n_2)\\&-(n_2+1)f_2^2(n_1,n_2+1)+h_1h_2+h_1h_3+h_2h_3,\end{aligned}$$

$$\begin{aligned}x_3=&(n_1+1)f_1^2(n_1+1,n_2)h_3+\\&(n_2+1)f_2^2(n_1.n_2+1)h_1-h_1h_2h_3,\end{aligned}$$

$$\begin{aligned}&|\phi_l\rangle_{eff}=a_l(n_1,n_2)|1,n_1,n_2\rangle+b_l(n_1,n_2)\\&|2,n_1+1,n_2\rangle+c_l(n_1,n_2)|3,n_1+1,n_2+1\rangle,\end{aligned}\quad (7a)$$

$$\begin{pmatrix}h_1-\mu_L & \sqrt{n_1+1}f_1(n_1+1,n_2) & 0\\ \sqrt{n_1+1}f_1(n_1+1,n_2) & h_2-\mu_L & \sqrt{n_2+1}f_2(n_1,n_2+1)\\ 0 & \sqrt{n_2+1}f_2(n_1,n_2+1) & h_3-\mu_L\end{pmatrix}\begin{pmatrix}a_l\\b_l\\c_l\end{pmatrix}=\mu_L\begin{pmatrix}a_l\\b_l\\c_l\end{pmatrix},\quad (8a)$$

with the following expansion coefficients:

$$a_l(n_1,n_2)=\frac{\sqrt{n_1+1}f_1(n_1+1,n_2)}{\mu_L(n_1,n_2)-h_1}b_l(n_1,n_2),$$

$$b_l(n_1,n_2)=\left(1+\frac{(n_1+1)f_1^2(n_1+1,n_2)}{(\mu_L(n_1,n_2)-h_1)^2}+\frac{(n_2+1)f_2^2(n_1,n_2+1)}{(\mu_L(n_1,n_2)-h_3)^2}\right)^{-\frac{1}{2}},$$

$$c_l(n_1,n_2)=\frac{\sqrt{n_2+1}f_2(n_1,n_2+1)}{\mu_L(n_1,n_2)-h_3}b_l(n_1,n_2),\quad (9a)$$

Now, we consider the eigenvalue equation for $\hat{H}$, which is defined by Eq. (23) in the form:

$$\hat{H}|\phi_L\rangle=\mu_L|\phi_L\rangle,\qquad L=1,2,3\quad (10a)$$

$$|\phi_L\rangle=U_1^\dagger U_2^\dagger U_3^\dagger|\phi_L\rangle_{eff},\quad (11a)$$

$$\begin{aligned}|\phi_L\rangle\approx&1-\varepsilon_1\left(\hat{a}^\dagger\hat{\sigma}_{12}-\hat{a}\hat{\sigma}_{21}\right)\\&-\varepsilon_2\left(\hat{b}^\dagger\hat{\sigma}_{23}-\hat{b}\hat{\sigma}_{32}\right)|\phi_l\rangle_{eff},=\\&a_l(n_1,n_2)|1,n_1,n_2\rangle+b_l(n_1,n_2)|2,n_1+1,n_2\rangle\\&+c_l(n_1,n_2)|3,n_1+1,n_2+1\rangle\end{aligned}$$

$$-\varepsilon_1\sqrt{n_1+2}b_l(n_1,n_2)|1,n_1+2,n_2\rangle$$
$$+\varepsilon_1\sqrt{n_1}a_l(n_1,n_2)|2,n_1-1,n_2\rangle$$
$$-\varepsilon_2\sqrt{n_2+2}c_l(n_1,n_2)|2,n_1+1,n_2+2\rangle$$
$$+\varepsilon_2\sqrt{n_2}bl(n_1,n_2)|3,n_1+1,n_2+1\rangle, \quad (12a)$$

$$\lambda_l(n_1,n_2)=\langle\phi_L|\psi(0)\rangle=a_l(n_1,n_2)P_{n_1}P_{n_2} -\varepsilon_1 b_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2}, \quad (13a)$$

The state vector of the system, at any time, up to the first-order approximation may be obtained in the following form:

$$|\psi(t)\rangle=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(A_{n_1.n_2}(t)|1,n_1,n_2\rangle +B_{n_1+1.n_2}(t)|2,n_1+1\rangle +C_{n_1.n_2+1}(t)|3,n_1+1,n_2+1\rangle, \quad (14a)$$

$$A_{n_1.n_2}(t)=\sum_{l=1}^{3} e^{-i\mu_L(n_1,n_2)t}(a_l(n_1,n_2)^2P_{n_1}P_{n_2} -\varepsilon_1 b_l(n_1,n_2)a_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2} -e^{-i\mu_L(n_1-2,n_2)t}\varepsilon_1 b_l(n_1-2,n_2)a_l(n_1-2,n_2)\sqrt{n_1}P_{n_1-2}P_{n_2},$$
(15a)

$$B_{n_1+1.n_2}(t)= \sum_{l=1}^{3} e^{-i\mu_L(n_1,n_2)t}(a_l(n_1,n_2)b_l(n_1,n_2)P_{n_1}P_{n_2} -\varepsilon_1 b_l(n_1,n_2)^2\sqrt{n_1+2}P_{n_1+2}P_{n_2} - e^{-i\mu_L(n_1+2,n_2)t}\varepsilon_2 a_l(n_1,n_2-2)c_l(n_1,n_2-2)\sqrt{n_2}P_{n_1}P_{n_2-2} +e^{-i\mu_L(n_1+2,n_2)t}\varepsilon_2\sqrt{n_2+2}a_l(n_1+2,n_2)^2P_{n_1}P_{n_2+2}, \quad (16a)$$

$$C_{n_1.n_2+1}(t)= \sum_{l=1}^{3} e^{-i\mu_L(n_1,n_2)t}(a_l(n_1,n_2)c_l(n_1,n_2)P_{n_1}P_{n_2} -\varepsilon_1 b_l(n_1,n_2)c_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2}) +e^{-i\mu_L(n_1,n_2+2)t}\varepsilon_2\sqrt{n_2+2}a_l(n_1,n_2+2)^2\, P_{n_1}P_{n_2+2}, \quad (17a)$$

Similarly, for $V$ −type atom we have,

$$\hat{H}_{eff}=\omega_f\hat{n}_1+\omega_{f'}\hat{n}_2+\tfrac{1}{3}\omega_f\hat{\sigma}_z^{13}+\tfrac{1}{3}\omega_{f'}\hat{\sigma}_z^{23}+ +\frac{(2\Delta_{13}-\Delta_{23})}{3}\hat{\sigma}_z^{13}+\frac{(2\Delta_{23}-\Delta_{13})}{3}\hat{\sigma}_z^{23} +\tfrac{1}{3}(E_1+E_2+E_3)(\hat{\sigma}_{11}+\hat{\sigma}_{22}+\hat{\sigma}_{33}) +\varepsilon_1 g_1(\hat{n}_1\hat{\sigma}_z^{13}-\hat{\sigma}_{33})+\varepsilon_2 g_2(\hat{n}_2\hat{\sigma}_z^{23}-\hat{\sigma}_{33}) +\hat{a}g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{13}+g_1 f_1(\hat{n}_1,\hat{n}_2)\hat{a}^\dagger\hat{\sigma}_{31} +\hat{b}g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{\sigma}_{23}+g_2 f_2(\hat{n}_1,\hat{n}_2)\hat{b}^\dagger\hat{\sigma}_{32},$$

$$\Delta_{12}=E_1-E_2-\omega_f,\quad \Delta_{23}=E_2-E_3-\omega_{f'},$$

$$\hat{N}''=\omega_f\hat{n}_1+\omega_{f'}\hat{n}_2+\tfrac{1}{3}\omega_f\hat{\sigma}_z^{13}+\tfrac{1}{3}\omega_{f'}\hat{\sigma}_z^{23}, \quad (18a)$$

Hence, for a fixed quantum number $N = n + 1$,and in unpaired atom-field states, $|1,n_1,n_2\rangle \equiv |1\rangle \otimes |n_1,n_2\rangle, \frac{1}{\sqrt{2}}(|2,n_1+1,n_2\rangle + |2,n_1,n_2+1\rangle) \equiv \frac{1}{\sqrt{2}}|2\rangle \otimes (|n_1+1+n_2+1\rangle), |3,n_1,n_2\rangle \equiv |3\rangle \otimes |n_1,n_2\rangle$, the eigenvalues can be estimated from the following matrix relation:

$$\begin{vmatrix} h_1-\mu_L & 0 & \frac{1}{\sqrt{2}}\sqrt{n_2+1}f_2(n_1,n_2+1) \\ 0 & h_2-\mu_L & \frac{1}{\sqrt{2}}\sqrt{n_1+1}f_1(n_1+1,n_2) \\ \frac{1}{\sqrt{2}}\sqrt{n_2+1}f_2(n_1,n_2+1) & \frac{1}{\sqrt{2}}\sqrt{n_1+1}f_1(n_1+1,n_2) & h_3-\mu_L \end{vmatrix}=0 \quad (19a)$$

where we have defined:

$$h_1=\omega_f n_1+\omega_{f'}\left(n_2+\tfrac{1}{3}\right)+\varepsilon_1 g_1 n_1+\frac{(2\Delta_{13}+\Delta_{23})}{3},$$

$$h_2=\omega_f n_1+\omega_{f'}\left(n_2+\tfrac{1}{3}\right)-\varepsilon_1 g_1-\varepsilon_1 g_1 n_1+\frac{(\Delta_{23}-\Delta_{13})}{3},$$

$$h_3=\omega_f\left(n_1+\tfrac{1}{2}\right)+\omega_{f'}\left(n_2+\tfrac{1}{2}\right) -\varepsilon_2 g_2-\varepsilon_2 g_2 n_2-\frac{(\Delta_{12}+\Delta_{23})}{3}, \quad (20a)$$

Ding algebraic calculations, the above equation is reduced to the following algebraic cubic equation:

$$\mu_L^3+x_1\mu_L^2+x_2\mu_L+x_3=0, \qquad L=1,2,3 \quad (21a)$$

where $\mu_L$ is the eigenvalue that corresponds to $L^{th}$ eigenvector of $\hat{H}_{eff}$ and

$$x_1=-(h_1+h_2+h_3),$$

$$x_2=-(n_1+1)f_1^2(n_1+1,n_2)-(n_2+1)f_2^2(n_1,n_2+1) +h_1h_2+h_1h_3+h_2h_3,$$

$$x_3=(n_1+1)f_1^2(n_1+1,n_2)h_3+(n_2+1)f_2^2(n_1,n_2+1)h_1 -h_1h_2h_3, \quad (22a)$$

$$|\phi_l\rangle_{eff}=a_l(n_1,n_2)|1,n_1,n_2\rangle+b_l(n_1,n_2)\frac{1}{\sqrt{2}} (|2,n_1+1,n_2\rangle+|2,n_1,n_2+1\rangle) +c_l(n_1,n_2)|3,n_1,n_2\rangle \quad (23a)$$

$$\begin{pmatrix} h_1-\mu_L & 0 & \frac{1}{\sqrt{2}}\sqrt{n_2+1}f_2(n_1,n_2+1) \\ 0 & h_2-\mu_L & \frac{1}{\sqrt{2}}\sqrt{n_1+1}f_1(n_1+1,n_2) \\ \frac{1}{\sqrt{2}}\sqrt{n_2+1}f_2(n_1,n_2+1) & \frac{1}{\sqrt{2}}\sqrt{n_1+1}f_1(n_1+1,n_2) & h_3-\mu_L \end{pmatrix} \begin{pmatrix} a_l \\ b_l \\ c_l \end{pmatrix} = \mu_L \begin{pmatrix} a_l \\ b_l \\ c_l \end{pmatrix}, \tag{24a}$$

with the following expansion coefficients,

$$a_l(n_1,n_2)=\frac{\sqrt{n_1+1}f_1(n_1+1,n_2)}{\sqrt{2}(\mu_L(n_1,n_2)-h_1)}b_l(n_1,n_2),$$

$$b_l(n_1,n_2)=\left(1+\frac{(n_1+1)f_1^2(n_1+1,n_2)}{2(\mu_L(n_1,n_2)-h_1)^2}+\frac{(n_2+1)f_2^2(n_1,n_2+1)}{2(\mu_L(n_1,n_2)-h_3)^2}\right)^{-\frac{1}{2}},$$

$$c_l(n_1,n_2)=\frac{\sqrt{n_2+1}f_2(n_1,n_2+1)}{\sqrt{2}(\mu_L(n_1,n_2)-h_3)}b_l(n_1,n_2). \tag{25a}$$

Now, we consider the eigenvalue equation for $\hat{H}$, which is defined by equation (14) in the form:

$$\hat{H}|\phi_L\rangle=\mu_L|\phi_L\rangle, \quad L=1,2,3, \tag{26a}$$

$$|\phi_L\rangle=U_1^\dagger U_2^\dagger U_3^\dagger|\phi_L\rangle_{eff}, \tag{27a}$$

$$\begin{aligned}&\approx 1-\varepsilon_1(\hat{a}^\dagger\hat{\sigma}_{12}-\hat{a}\hat{\sigma}_{21})-\varepsilon_2(\hat{b}^\dagger\hat{\sigma}_{32}-\hat{b}\hat{\sigma}_{23})|\phi_l\rangle_{eff} \\ &=a_l(n_1,n_2)|1,n_1,n_2\rangle+b_l(n_1,n_2)|2,n_1+1,n_2\rangle+c_l(n_1,n_2)|3,n_1+1,n_2+1\rangle \\ &-\varepsilon_1\sqrt{n_1+2}b_l(n_1,n_2)|1,n_1+2,n_2\rangle+\varepsilon_1\sqrt{n_1}a_l(n_1,n_2)|2,n_1-1,n_2\rangle \\ &-\varepsilon_2\sqrt{n_2+2}c_l(n_1,n_2)|2,n_1+1,n_2+2\rangle+\varepsilon_2\sqrt{n_2}bl(n_1,n_2)|3,n_1+1,n_2+1\rangle,\end{aligned} \tag{28a}$$

$$\lambda_l(n_1,n_2)=\langle\phi_L|\psi(0)\rangle=a_l(n_1,n_2)P_{n_1}P_{n_2}-\varepsilon_1 b_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2}, \tag{29a}$$

The state vector of the system at any time $t$ up to the first-order approximation may be obtained in the following form:

$$|\psi(t)\rangle=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(A_{n_1,n_2}(t)|1,n_1,n_2\rangle+B_{n_1+1,n_2}(t)|2,n_1+1,n_2\rangle+C_{n_1,n_2+1}(t)|3,n_1+1,n_2+1\rangle), \tag{30a}$$

$$\begin{aligned}A_{n_1,n_2}(t)=\sum_{l=1}^{3}& e^{-i\mu_L(n_1,n_2)t}(a_l(n_1,n_2)^2P_{n_1}P_{n_2}-\varepsilon_1 b_l(n_1,n_2)a_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2} \\ &-e^{-i\mu_L(n_1-2,n_2)t}\varepsilon_1 b_l(n_1-2,n_2)a_l(n_1-2,n_2)\sqrt{n_1}P_{n_1-2}P_{n_2},\end{aligned} \tag{31a}$$

$$\begin{aligned}B_{n_1+1,n_2}(t)=\sum_{l=1}^{3}& e^{-i\mu_L(n_1,n_2)t}(a_l(n_1,n_2)b_l(n_1,n_2)P_{n_1}P_{n_2}-\varepsilon_1 b_l(n_1,n_2)^2\sqrt{n_1+2}P_{n_1+2}P_{n_2} \\ &-e^{-i\mu_L(n_1+2,n_2)t}\varepsilon_2 a_l(n_1,n_2-2)c_l(n_1,n_2-2)\sqrt{n_2}P_{n_1}P_{n_2-2} \\ &+e^{-i\mu_L(n_1+2,n_2)t}\varepsilon_2\sqrt{n_2+2}a_l(n_1+2,n_2)^2P_{n_1}P_{n_2+2},\end{aligned} \tag{32a}$$

$$\begin{aligned}C_{n_1,n_2+1}(t)=&\sum_{l=1}^{3} e^{-i\mu_L(n_1,n_2)t}a_l(n_1,n_2)c_l(n_1,n_2)P_{n_1}P_{n_2} \\ &-\varepsilon_1 b_l(n_1,n_2)c_l(n_1,n_2)\sqrt{n_1+2}P_{n_1+2}P_{n_2}) \\ &+e^{-i\mu_L(n_1,n_2+2)t}\varepsilon_2\sqrt{n_2+2}a_l(n_1,n_2+2)^2P_{n_1}P_{n_2+2}.\end{aligned} \tag{33a}$$

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