Investigation of the Effects of Kerr-like Environments on Entanglement between Two Single-atom Cavities

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ABSTRACT— In this research, we investigate the effects of Kerr-like environments on the entanglement between two cavities. Each cavity contains a multi-photon field and a two-level atom. We use basis states to examine entanglement, where there are two excitations in the total system (atom + field). Since we consider the cavity environments to be Kerr-like, we employ the deformed Jaynes–Cummings model (DJCM). Due to the physical model used, we assume the deformation function depends on the photon number. It is assumed that there is no direct coupling between the two cavities or between the two atoms, and each atom interacts locally with its own field. We will show that the presence or absence of generalized deformation will affect entanglement both qualitatively and quantitatively. The criterion used to examine entanglement is the concurrence parameter. We will show that when the excitation is equally shared between the atom and the field, maximum entanglement between them occurs. Furthermore, when the excitation is concentrated solely on the atom or solely on the field, the entanglement between the atom and the field will vanish. In addition, we will show that when deformation is present, the entanglement time of the system increases. However, for some initial conditions, deformation causes sudden death of entanglement, resulting in entanglement instability, while for sufficiently large values of deformation, the system reaches a steady state. We will also show that the Kerr-like nature of the environment increases entanglement.

KEYWORDS: Concurrence parameter, Coupling Constant, Deformed Jaynes-Cummings Model, Detuning, Entanglement

I. INTRODUCTION

Quantum entanglement is a fundamental characteristic of quantum mechanics that establishes and delineates the essential difference between classical and quantum physics. The authors' motivation for examining this paper is as follows. First, entangled states represent a type of non-local quantum correlation between subsystems and have numerous applications in various areas of physics, including quantum information transfer [1]-[3], quantum computing [4], quantum teleportation [5], cryptography [6], and dense coding [7]. One of the models widely used in quantum optics and entanglement studies is the Jaynes-Cummings model (JCM), which was introduced in 1963 [8]. This model was initially employed to investigate the interaction between a two-level atom and a single-mode quantum field [9], [10]. It predicts interesting phenomena such as damping and revival in population inversion [11], [12], entanglement [13]-[15], and squeezing [16]. In recent years, this model has been utilized to study multi-level atoms [17], [18], multi-mode fields [19], [20], and deformed fields [21] in quantum optics and lasers. Recently, deformed Jaynes-Cummings models have been used for nonlinear atom-field interactions [22]-[24]. For the first time, Buzek and Sukumar examined the nonlinear interaction of a quantum field with a two-level atom using the deformed Jaynes-Cummings model [25], [26]. The nonlinear model induces entanglement, which can be exploited in quantum communication and information. Generalized Jaynes-Cummings

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models—such as f-deformed, q-deformed, and ζ-deformed variants—have been investigated in references [27,28], [29,30], and [31], respectively. Recently, the quantum dynamics of the interaction between a single-mode field and a two-level atom in the presence of the Lamb shift has been studied [32], [33].

The authors' motivation for this research is as follows:

1. Quantum entanglement, as mentioned above, has numerous applications in various fields of physics (quantum information, quantum computing, quantum teleportation, cryptography).

2. As mentioned above, although the non-deformed Jaynes-Cummings model (JCM) is a highly effective model and can describe many experimental results of atom-field interactions, today its generalized models, such as the f-deformed Jaynes-Cummings model (f-DJCM), have attracted considerable attention. For example, this model can be used for atom-field interaction with intensity-dependent coupling. Furthermore, this model can also be employed for atom-field interaction in Kerr-like media ($f(m)=\sqrt{m}$)

3. Although in the model used in this work there is no direct coupling between the two cavities or the two atoms, the initial state of the system is entangled. Consequently, during the time evolution, the local interaction of each atom with its own field causes the transfer and redistribution of entanglement among the subsystems. Investigating this phenomenon (in the absence of direct coupling) is fundamentally important for understanding the stability of entanglement in quantum networks.

4. The topic of the paper is scientifically and practically interesting and addresses one of the active areas in quantum optics and quantum information. Examining the role of detuning and the Kerr-like medium in the stability or sudden death of entanglement (ESD) is a very significant idea.

In this research, we investigate the entanglement between two deformed, single-atom, multi-photon cavities. The model used is the deformed Jaynes-Cummings model (DJCM). In the model under discussion, it is assumed that atom A interacts only with field $\hat{B}_1$ and atom B only with field $\hat{B}_2$; therefore, entanglement is expected to be local as well. Nevertheless, we will show that under certain initial conditions, entanglement emerges between atoms and the fields of adjacent cavities. We will demonstrate that using the deformed *f*-Jaynes-Cummings model, the entanglement between the two atoms A and B can be controlled via detuning parameters, the number of photons in the cavities, and the deformation function. Detuning here refers to the difference between the atomic transition frequency and the cavity field frequency. For the case where the atomic transition frequencies and the field frequencies are not in resonance (i.e., detuning is present), we calculate all pairwise entanglements between the qubit pairs. We also show that maximum atom–field entanglement occurs when the excitation is equally shared between the atom and the field. Furthermore, when the excitation is initially localized solely in the atom or solely in the field, the atom–field entanglement vanishes. Due to the deformation function ($f(m)=\sqrt{m}$), this model can be experimentally implemented to investigate entanglement between two cavities, each containing a two-level atom and an un-deformed multi-photon field, in Kerr-like media. In this model, it is assumed that each atom interacts locally with its own cavity field.

II. HAMILTONIAN AND ENERGY EIGENVALUES

In this part, we first introduce the Hamiltonian of the system, then obtain its eigen-functions and eigenvalues, and using them, we examine the time evolution of entanglement.

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A. *Deformed Jaynes-Cummings Hamiltonian*

In this part, we consider a system consisting of two separate cavities as shown in Fig. 1, each containing a two-level atom and a multi-photon field, and we take the cavity environment to be Kerr-like. In general, the two cavities are non-identical, and the frequencies of the two cavities, the coupling strengths between the atoms and the cavity modes, as well as the number of photons in the cavities, may differ from one another.

$$\hat{H} = \hat{H}_{field} + \hat{H}_{atom} + \hat{H}_{\text{int}} \tag{1}$$

where, $\hat{H}_{field}$ is the Hamiltonian of the cavity modes. If we assume that the resonance frequency of the two cavities is ω_1 and ω_2, we can write:

$$\hat{H}_{field} = \hbar\omega_1 \hat{B}_1^{\dagger}\hat{B}_1 + \hbar\omega_2 \hat{B}_2^{\dagger}\hat{B}_2 \tag{2}$$

Also, $\hat{H}_{atom}$ is the Hamiltonian of the atoms, and if we assume they have a transition frequency ω_0, it can be written as:

$$\hat{H}_{atom} = \hbar\omega_0 S_A^z + \hbar\omega_0 S_B^z \tag{3}$$

We also assume that the interaction of each atom with the field inside the cavity occurs through multiphoton transitions:

$$\hat{H}_{\text{int}} = \hbar g_1\left((\hat{B}_1^{m_1})^{\dagger}\hat{S}_A^{-} + \hat{B}_1^{m_1}\hat{S}_A^{+}\right) + \hbar g_2\left((\hat{B}_2^{m_2})^{\dagger}\hat{S}_B^{-} + \hat{B}_2^{m_2}\hat{S}_B^{+}\right) \tag{4}$$

Furthermore, m_1 and m_2 are the absorbing or emitting photon numbers per atomic transition photon numbers per atomic transition in cavity 1 and cavity 2, respectively. Here, the parameters g_1 and g_2 are the coupling constants between the atoms and the cavity modes, which we assume to be real without loss of generality. Also, $\hat{S}_i^{+}$ and $\hat{S}_i^{-}$ is the raising and lowering operator for the *i*-th atom, respectively. Furthermore, here $\hat{B}_i^{\dagger}$ and $\hat{B}_i$ are the deformed creation and annihilation operators corresponding to the *i*-th cavity, and it can be written as:

$$\begin{aligned} &\hat{B}_1 = \hat{b}_1 f_1(\hat{m}_1), \hat{B}_1^{\dagger} = f_1(\hat{m}_1)\hat{b}_1^{\dagger} \\ &\hat{B}_2 = \hat{b}_2 f_2(\hat{m}_2), \hat{B}_2^{\dagger} = f_2(\hat{m}_2)\hat{b}_2^{\dagger} \end{aligned} \tag{5}$$

where $\hat{m}_i = \hat{b}_i^{\dagger}\hat{b}_i$ is the number operator and $f_i(\hat{m}_i)$ is an arbitrary real function of the number operator. Regarding deformed fields, for the commutation relation we can write:

$$\left[\hat{B}_i, \hat{B}_j^{\dagger}\right] = \delta_{ij}\left[(m_i+1) f_i^2(m_i+1) - m_i f_i^2(m_i)\right] \tag{6}$$

For the case of un-deformed fields ($f_i(\hat{m}_i)=1$), we have:

$$\left[\hat{b}_i, \hat{b}_j^{\dagger}\right] = \delta_{ij} \tag{7}$$

The commutation relation is the usual one. The Hamiltonian model used in the above equations is the deformed Jaynes-Cummings model (DJCM). For the case of un-deformed fields ($f_i(\hat{m}_i)=1$), we have:

$$\hat{H}_{field} = \hbar\omega_1 \hat{b}_1^{\dagger}\hat{b}_1 + \hbar\omega_2 \hat{b}_2^{\dagger}\hat{b}_2 \tag{8}$$

$$\hat{H}_{\text{int}} = \hbar g_1\left((\hat{b}_1^{m_1})^{\dagger}\hat{S}_A^{-} + \hat{b}_1^{m_1}\hat{S}_A^{+}\right) + \hbar g_2\left((\hat{b}_2^{m_2})^{\dagger}\hat{S}_B^{-} + \hat{b}_2^{m_2}\hat{S}_B^{+}\right) \tag{9}$$

It reduces to the un-deformed Jaynes-Cummings model (JCM). In previous works [34-35], it has been assumed that the cavities are in resonance and identical, and the fields have also been assumed to be un-deformed $\left[\omega_1 = \omega_2 = \omega_0,\ g_1 = g_2,\ \text{and}\ f_1(m_1) = f_2(m_2) = 1\right]$.

Here, because we have considered the environment to be cavity-like, we have used the deformed Jaynes-Cummings model. In this case, subsystems 1 and 2 become indistinguishable, which may lead to greater stability of entanglement. We will also show that the presence of asymmetric cavities and the presence of nonlinearity are factors that can be used to control entanglement. Moreover, the model in which $\omega_1 \neq \omega_2 \neq \omega_0$ and $g_1 \neq g_2$ is

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assumed is also a more realistic model from an experimental perspective. For example, unequal coupling constants occur when atoms are not located at the same position inside the cavities. Furthermore, we will demonstrate that asymmetric cavities destroy the simple entanglement transfer rules reported by Yonac, *et al*. in reference [35].

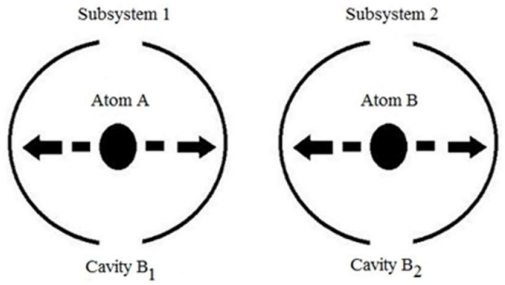


Fig. 1. Two separate cavities, each containing a two-level atom and a deformed multi-photon field, where the atoms and the fields interact locally

B. *Ground States of Energy*

In this part, we assume that there is an initial entanglement between atoms A and B, and then we investigate the time evolution of the entanglement. We also assume that the two cavities have no energy exchange with their surrounding environment, and each atom only interacts locally with the cavity field in which it is placed. We consider the eigen-functions of the non-interacting Hamiltonian as the product of the states of the atoms and the fields. The energy eigenstates for a two-level atom are the two states: excited $|+\rangle$ and ground $|-\rangle$, and we can write:

$$\hat{S}_z|+\rangle=\frac{1}{2}|+\rangle,\hat{S}_z|-\rangle=-\frac{1}{2}|-\rangle \quad (10)$$

Cavity energy states are represented by the Fock state $|m\rangle$, where *n* is the quantum number corresponding to the cavity modes. Here, following the approach of Yonac, *et al.* [34], two types of initial entanglement can be considered. In the first type, it is assumed that the system is in a state with anti-parallel spins.

$$|\xi\rangle=\cos(\theta)|+,-,m_1,m_2\rangle+e^{i\alpha}\sin(\theta)|-,+,m_1,m_2\rangle \quad (11)$$

In Eq. (11), $\cos^2(\theta)$ is the probability that the system is initially in state $|+,-,m_1,m_2\rangle$, and $\sin^2(\theta)$ is the probability that the system is initially in state $|-,+,m_1,m_2\rangle$. Since we want a phase difference between these two states, we have introduced the phase factor $e^{i\alpha}$ in the second term. In the second type, it is assumed that the system is in a state with parallel spins.

$$|\chi\rangle=\cos\theta|+,+,m_1,m_2\rangle+e^{i\alpha}\sin\theta|-,-,m_1,m_2\rangle \quad (12)$$

Here, $|+,-,i,j\rangle$ indicates that the A atom is in the excited state and the B atom is in the ground state. Also, *i* and *j* represent the number of photons in the B_1 and B_2 fields. In Section □, it is assumed that the basis vectors of the system are covered by the following four state vectors:

$$\begin{aligned}|\chi_1\rangle&=|+,+,m_1,m_2\rangle\\|\chi_2\rangle&=|+,-,m_1,2m_2\rangle\\|\chi_3\rangle&=|-,+,2m_1,m_2\rangle\\|\chi_4\rangle&=|-,-,2m_1,2m_2\rangle\end{aligned} \quad (13)$$

In these basis vectors, there are two excitations in the entire atom-field system. For example, the $|\chi_1\rangle=|+,+,m_1,m_2\rangle$ state is a state where both excitations are associated with the atoms, because we have assumed that initially, in cavities A and B, there are m_1 and m_2 photons, respectively. Or the $|\chi_2\rangle=|+,-,m_1,2m_2\rangle$ state, where one of the excitations is associated with atom A and the other with the $B_2(m_2\rightarrow 2m_2)$ cavity field.

We also investigate hybrid states of the $\sum_{i=1}^{4}\chi_i\rangle/2$ type, in which all four states of Eq. (9) have equal weight, and show that in this case, although the amount of entanglement decreases, its stability increases. To examine the amount of entanglement, we use the concurrence parameter introduced by Wootters [36]. Concurrence is defined by the following relation:

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$$C = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right) \quad (14)$$

where, λ_i is a special function of the density operator $\hat{\rho}' = \hat{\rho}\left(\hat{\sigma}_y \otimes \hat{\sigma}_y\right)\hat{\rho}^*\left(\hat{\sigma}_y \otimes \hat{\sigma}_y\right)$, and $\hat{\sigma}_y$ is the Pauli matrix in the direction of y. Maximum entanglement occurs in state C=1 and C=0 represents a state in which there is no entanglement between atoms and fields. In the case where C>0 is present, entanglement definitely exists. The concurrence for $|\chi\rangle$ states is obtained from the following relation [37]:

$$C_{12} = 2\sin(\theta)\cos(\theta) \quad (15)$$

Equation (10**)** shows that the maximum entanglement is obtained for $\theta=\pi/4$. In the case where the system is in the initial state $|\chi\rangle$, the sum of squares of non-local entanglement satisfies the following relation [37]:

$$0 \le C_{AB}^2 + C_{AA_2}^2 + C_{A_1B}^2 + C_{A_1A_2}^2 \le C_{12}^2 \quad (16)$$

III. INVESTIGATION OF ENTANGLEMENT

In this section, we investigate the time evolution of entanglement. The entanglement properties in the case where there is no detuning and the atom-field coupling constants are equal have been studied in [34-35]. Here, we assume that there is detuning in the system, and also that the atom-field coupling constants for the two subsystems are not identical, and furthermore, both fields are deformed.

A. *Solving the Dynamical Equations*

To solve the dynamical equations, we introduce the wave function of the system as follows:

$$\begin{aligned}|\xi(t)\rangle &= s_1(t)|+,+,m_1,m_2\rangle + s_2(t)|+,-,m_1,2m_2\rangle \\ &+ s_3(t)|-,+,2m_1,m_2\rangle + s_4(t)|-,-,2m_1,2m_2\rangle \\ &+ s_5(t)|-,-,m_1,m_2\rangle \end{aligned} \quad (17)$$

The above relation is the linear combination of the attainable states of the atoms and cavity fields at time t. The coefficients $s_i(t)$ represent the probability amplitudes for the i-th state at time t. Using the Schrödinger equation

$$i\hbar\frac{d}{dt}|\xi(t)\rangle = \hat{H}|\xi(t)\rangle \quad (18)$$

We arrive at the following differential equations for the probability amplitudes (proof in Appendix A).

$$\dot{\tilde{s}}_1 = -i\left(\Delta_1' + \Delta_2'\right)\tilde{s}_1 - i\left(g_2'\tilde{s}_2 + g_1'\tilde{s}_3\right) \quad (19)$$

$$\dot{\tilde{s}}_2 = -i\left(\Delta_1' - \Delta_2'\right)\tilde{s}_2 - i\left(g_2'\tilde{s}_2 + g_1'\tilde{s}_4\right) \quad (20)$$

$$\dot{\tilde{s}}_3 = -i\left(\Delta_1' - \Delta_2'\right)\tilde{s}_3 - i\left(g_1'\tilde{s}_1 + g_2'\tilde{s}_4\right) \quad (21)$$

$$\dot{\tilde{s}}_4 = -i\left(\Delta_1' - \Delta_2'\right)\tilde{s}_2 - i\left(g_1'\tilde{s}_2 + g_2'\tilde{s}_3\right) \quad (22)$$

We have defined here:

$$\Delta_j' = \frac{\omega_0 - 2n_j f_j^2(2m_j)\omega_j}{2} \quad (23)$$

$$g_1' = g_1\sqrt{\frac{(2n_1)!}{n_1!}}f_1(2m_1) \quad (24)$$

$$g_2' = g_2\sqrt{\frac{(2n_2)!}{n_2!}}f_2(2m_2) \quad (25)$$

Here, Δ_j', g_1', and g_2' represents the generalized de-tunings and generalized couplings related to cavities 1 and 2. In fact, because we have assumed the field to be deformed, we are dealing with Δ_j', g_1', and g_2' quantities instead of Δ_j', g_1', and g_2' quantities. Here Δ_j is the detuning related to the j-th field, and $(\omega_0 - \omega_j)$ is the frequency difference between the atomic transition and the j-th field. Furthermore, to eliminate the rapidly varying terms, we have used the following slowly varying function:

$$\tilde{s}_i = s_i \exp\left\{-i\left(\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right)t\right\} \quad (26)$$

Equations (19)-(22) are a set of coupled equations with time-independent coefficients, which can be solved using matrix equations.

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Due to the complexity of the calculations, the solutions to these equations are obtained for two different cases. In the first case, we assume that the generalized de-tunings are nonzero but the generalized coupling constants are equal. In the second case, we assume the generalized de-tunings to be zero, while the generalized coupling constants are unequal. Instead of obtaining the solutions of the equations in terms of $\tilde{s}_i\,(i=1,2,3,4)$, we derive the solutions based on $s^{(i)}\,(i=1,2,3,4)$, which is a linear combination of the probability amplitudes and is defined by the following relations.

$$s^{(1)}(t)=\tilde{s}_1(t)+\tilde{s}_4(t) \tag{27}$$

$$s^{(2)}(t)=\tilde{s}_1(t)-\tilde{s}_4(t) \tag{28}$$

$$s^{(3)}(t)=\tilde{s}_2(t)+\tilde{s}_3(t) \tag{29}$$

$$s^{(4)}(t)=\tilde{s}_2(t)-\tilde{s}_3(t) \tag{30}$$

For the case where the generalized displacements are non-zero and the generalized coupling constants are equal, we have:

$$\begin{aligned} s^{(1)}(t)&=s^{(1)}(0)\cos(\Omega g t)-\\ &\frac{2i}{\Omega}\left[s^{(3)}(0)+\delta s^{(2)}(0)\right]\sin(\Omega g t)\end{aligned} \tag{31}$$

$$\begin{aligned} s^{(2)}(t)&=\frac{4}{\Omega^2}\left[s^{(2)}(0)-\delta s^{(3)}(0)\right]+\frac{4\delta}{\Omega^2}\left[s^{(3)}(0)+\right.\\ &\left.+\delta s^{(2)}(0)\right]\cos(\Omega g t)-\frac{2i\delta}{\Omega}s^{(1)}(0)\sin(\Omega g t)\end{aligned} \tag{32}$$

$$\begin{aligned} s^{(3)}(t)&=\frac{4}{\Omega^2}\left[\delta s^{(3)}(0)-s^{(2)}(0)\right]+\frac{4\delta}{\Omega^2}\left[s^{(3)}(0)\right.\\ &\left.+\delta s^{(2)}(0)\right]\cos(\Omega g t)-\frac{2i}{\Omega}s^{(1)}(0)\sin(\Omega g t)\end{aligned} \tag{33}$$

$$s^{(4)}(t)=s^{(4)}(0) \tag{34}$$

Here, for simplicity, we have assumed:

$$\Delta_1'=\Delta_2'=\Delta' \tag{35}$$

Also, Ω is the dimensionless Rabi frequency, which is defined by the following relation:

$$\Omega=2\left(1+\delta^2\right)^{\frac{1}{2}} \text{ and } \delta=\Delta'/g' \tag{36}$$

For the case where $\Delta_1'=\Delta_2'=0$ is present but $g_1'\neq g_2'$ is present, we can write:

$$\begin{aligned} s^{(1)}(t)&=s^{(1)}(0)\cos(2gt)-is^{(3)}(0)\sin(2gt)\\ s^{(2)}(t)&=s^{(2)}(0)\cos(2ut)+is^{(4)}(0)\sin(2ut)\\ s^{(3)}(t)&=s^{(3)}(0)\cos(2gt)-is^{(1)}(0)\sin(2gt)\\ s^{(4)}(t)&=s^{(4)}(0)\cos(2ut)+is^{(2)}(0)\sin(2ut)\end{aligned} \tag{37}$$

We have defined here $g=\frac{g_1'+g_2'}{2}$ and $u=\frac{g_1'-g_2'}{2}$.

B. *Time Evolution of Entanglement*

In this part, we consider the state of the system as follows:

$$\begin{aligned} |\xi(t)\rangle&=s_1(t)|+,+,m_1,m_2\rangle+s_2(t)|+,-,m_1,2m_2\rangle\\ &+s_3(t)|-,+,2m_1,m_2\rangle+s_4(t)|-,-,2m_1,2m_2\rangle\end{aligned} \tag{38}$$

In this case, regardless of what the expansion coefficients are, atom-atom entanglement (C_{AB}) does not occur. By taking the partial trace of the total density matrix over the field states, the reduced atomic density matrix is obtained. Therefore, it can be written as:

$$\rho_{AB}=Tr_{cavity}\rho=\sum_{i,j=0,1}\langle i|\langle j|\rho|j\rangle|i\rangle \tag{39}$$

where, $|i\rangle$ and $|j\rangle$ represents the states of the B_1 and B_2 fields, respectively. Using Eq. (24), it can be written as:

$$\begin{aligned} \rho_{AB}&=|s_1(t)|^2|+,+\rangle\langle+,+|+|s_2(t)|^2|+,-\rangle\langle+,-|\\ &+|s_3(t)|^2|-,+\rangle\langle-,+|+|s_4(t)|^2|-,-\rangle\langle-,-|\end{aligned} \tag{40}$$

In this case, the entanglement C_{AB}, $C_{B_1B_2}$, C_{AB_2}, and C_{BB_1} will always be zero. The only nonzero concurrences belong to the C_{AB_1} and C_{BB_2} terms. In other words, entanglement only occurs

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between atoms and the fields with which they interact. For local entanglements, we can write:

$$C_{AB_1}(t) = 2\left|\tilde{s}_1(t)\tilde{s}_3^*(t) + \tilde{s}_2(t)\tilde{s}_4^*(t)\right| \quad (41)$$

$$C_{BB_2}(t) = 2\left|\tilde{s}_1(t)\tilde{s}_2^*(t) + \tilde{s}_3(t)\tilde{s}_4^*(t)\right| \quad (42)$$

In Fig. 2(a), the time evolution diagram of C_{AB_1} for the initial state $|+,+,m_1,m_2\rangle$ and for the un-deformed state $\left(f_1(m_1)=f_2(m_2)=1\right)$ is plotted. It is observed that entanglement between an atom and the field occurs during the excitation transition stage between them. We also observe that when the excitation is equally divided between the atom and the field, maximum entanglement between the atom and the field occurs. Furthermore, when the excitation is concentrated solely on the atom or the field, the entanglement between the atom and the field will disappear. Also, at $t = n\pi/g\Omega$, $(n=1,2,3,...)$ times, there will be no entanglement between the atom and the field. Furthermore, we observe that when detuning exists, the entanglement time of the system increases. In fact, the presence of detuning will cause greater stability of entanglement. In the presence of detuning, at $t = 2n\pi/g\Omega$, $(n=0,1,2,...)$ times, the entanglement of the system will become zero. This is understandable given the fact that entanglement is a result of a superposition corresponding to the non-locality of energy. When the field frequency and the atomic transition frequency are not in resonance, the initial amount of local energy concentrated in the atom will not be completely transferred to the cavity field. In fact, part of the energy will not be absorbed by the cavity field and remains non-local, resulting in non-zero entanglement. In Fig. 2(b), the time evolution diagram of C_{AB_1} for the initial state $|+,+,m_1,m_2\rangle$ and for the deformed state $\left(m_1=m_2=30 \text{ and } f_1(m_1)=f_2(m_2)=\sqrt{m}\right)$ is plotted. In this case, as can be seen, the diagram becomes more oscillatory with respect to time, which is predictable given the non-Markovian nature of the environment. The physical interpretation of the obtained results is that for deformed fields, compared to non-deformed fields, the energy exchange between atoms and fields occurs more rapidly over equal time intervals. Given the non-Markovian nature of the environment for deformed fields, the obtained results are consistent with the physics of the problem. In fact, when the oscillation intensity of the entanglement measure (concurrence) C_{AB_1} increases for the case of deformed fields compared to non-deformed fields, it means that in non-Markovian environments, the energy exchange between atom A and field B_1 occurs more rapidly. Indeed, the Hamiltonian model introduced in Eq. (4) is a type of atom-field interaction model with intensity-dependent coupling, which itself strengthens the interaction between the atom and the field, and as a result, the energy exchange between the atom and the field occurs more quickly. In fact, the increase in the oscillation intensity of the concurrence parameter indicates the rapid growth of energy transfer between atoms and fields. On the other hand, in the deformed state, in both cases of zero detuning $(\Delta=0)$ and non-zero detuning $(\Delta\neq0)$, the time interval between two consecutive zero entanglements decreases. Also, because the parameters in the case of deformed fields are functions of the deformation parameter, it can be concluded that in both cases of zero detuning and non-zero detuning, the times at which entanglement becomes zero depend on the number of photons inside the environment. In fact, by using the number of photons inside the cavity, the time interval between two consecutive zero entanglements can be changed.

In solving the Eqs. (19)-(22), if we assume that the condition $(\Delta_1' \neq \Delta_2')$ is not used, the solutions will be very lengthy, and their numerical results have been plotted as graphs for two different cases $(\Delta_1'=2, \Delta_2'=1)$ and $(\Delta_1'=10, \Delta_2'=15)$. The graphs have been plotted for both deformed fields and non-deformed fields. We observe that in the case where the de-tunings between the two cavities are not equal, the entanglement will take the

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form of oscillatory functions, such that the greater the difference in detuning between the two cavities, the more oscillatory the entanglement will be. In other words, the greater the difference in detuning between the two cavities, the faster the energy exchange between atom A and the field occurs. Furthermore, the entanglement will be more oscillatory for the case of Kerr-like media. The above results are shown in the Figs. 3(a), 3(b), 4(a), and 4(b).

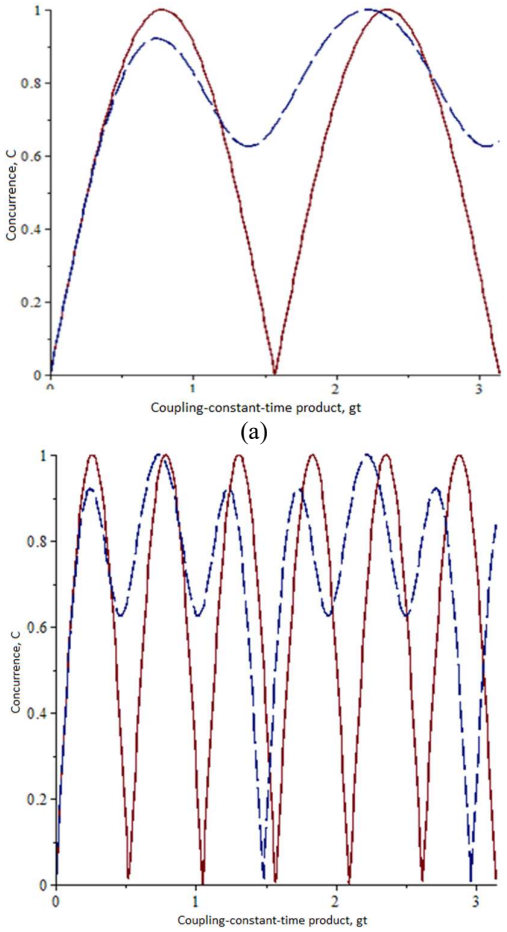


Fig. 2. (a) Time evolution of $C_{AB_1}(t)$ for the initial state $|+,+,m_1,m_2\rangle$. The solid line represents the case $\Delta'=0$, and the dashed line represents the case $\Delta'\neq 0$ for the un deformed case. Here, we have assumed $m_1=m_2=30$ and $f_1(m_1)=f_2(m_2)=1$, (b) Time evolution of $C_{AB_1}(t)$ for the initial state $|+,+,m_1,m_2\rangle$. The solid line represents the case $\Delta'=0$, and the dashed line represents the case $\Delta'\neq 0$ for the un deformed case. Here, we have assumed $m_1=m_2=30$ and $f_1(m_1)=f_2(m_2)=\sqrt{m}$.

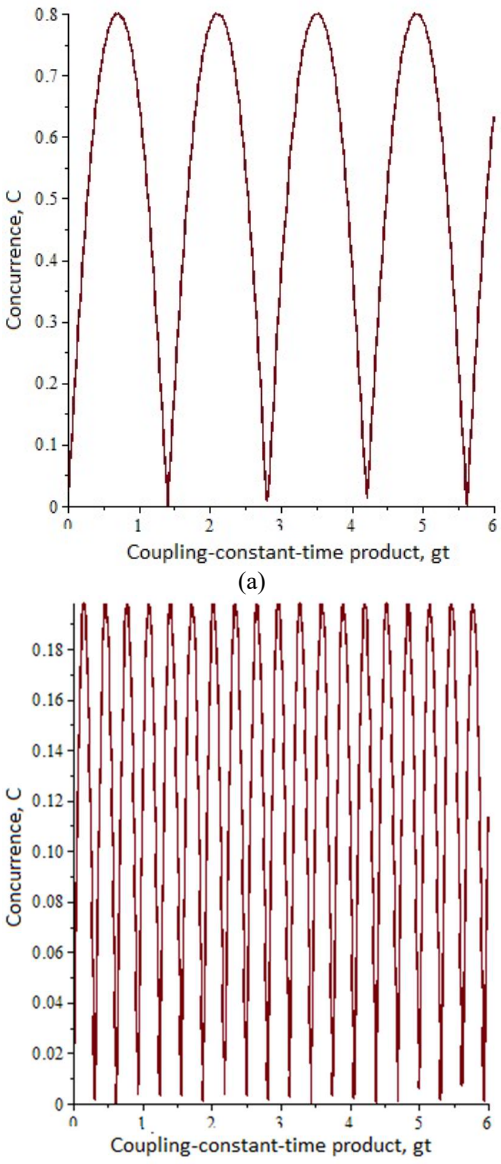


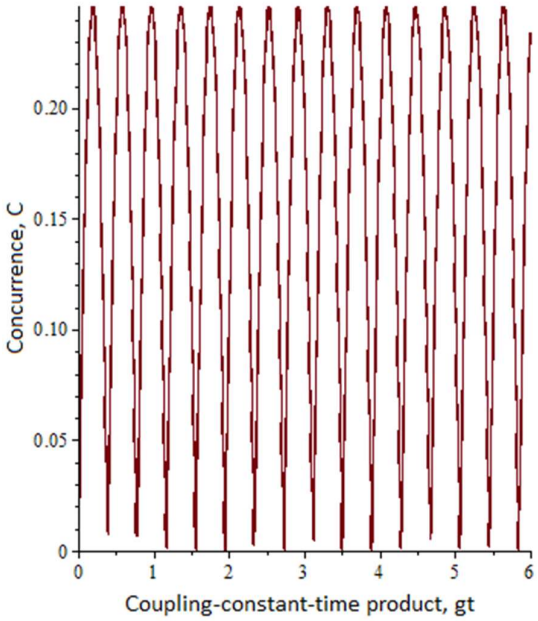
Fig. 3. (a) Time evolution of $C_{AB_1}(t)$ for the initial state $|+,+,m_1,m_2\rangle$. Here, we have assumed $m_1=m_2=30$, $f_1(m_1)=f_2(m_2)=1$, $\Delta_1'=2$, and $\Delta_2'=1$. (b) Time evolution of $C_{AB_1}(t)$ for the initial state $|+,+,m_1,m_2\rangle$. Here, we have assumed $m_1=m_2=30$, $f_1(m_1)=f_2(m_2)=1$, $\Delta_1'=10$, and $\Delta_2'=15$.

C. *Investigating Entanglement for the State* $|\chi_0\rangle=|-,-,m_1,m_2\rangle$

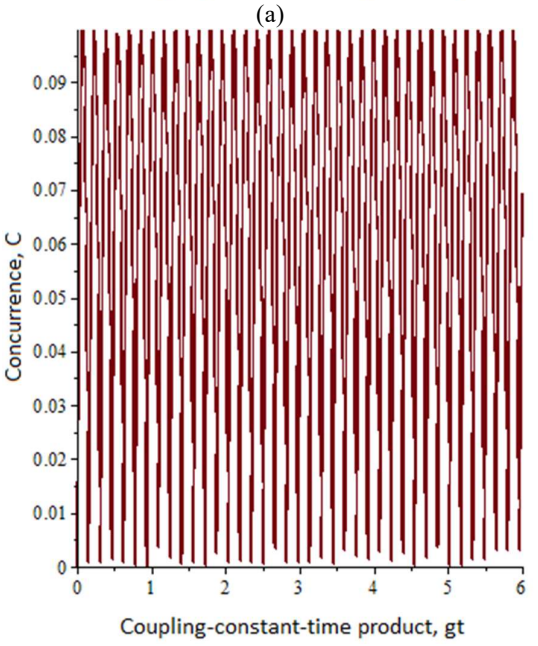
In this part, we examine entanglement for the case where the system is initially in the state $|\chi_0\rangle=|-,-,m_1,m_2\rangle$. In this case, the entanglements between the atoms and the fields

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of the two subsystems are obtained from the relations:



(a)



(b)

Fig. 4. (a) Time evolution of $C_{AB_1}(t)$ for the initial state $|+,+,m_1,m_2\rangle$. Here, we have assumed $m_1=m_2=30$, $f_1(m_1)=f_2(m_2)=\sqrt{m}$, $\Delta_1'=2$, and $\Delta_2'=1$. (b) Time evolution of $C_{AB_1}(t)$ for the initial state $|+,+,m_1,m_2\rangle$. Here, we have assumed $m_1=m_2=30$, $f_1(m_1)=f_2(m_2)=\sqrt{m}$, $\Delta_1'=10$, and $\Delta_2'=15$.

$$
\begin{aligned}
C_{B_1B_2} &= \max\left\{0,2\left(|s_4(t)||s_0(t)|-|s_2(t)||s_3(t)|\right)\right\}\\
C_{AB} &= \max\left\{0,2\left(|s_1(t)||s_0(t)|-|s_2(t)||s_3(t)|\right)\right\}\\
C_{B_1B} &= \max\left\{0,2\left(|s_3(t)||s_0(t)|-|s_1(t)||s_4(t)|\right)\right\}\\
C_{AB_2} &= \max\left\{0,2\left(|s_2(t)||s_0(t)|-|s_1(t)||s_4(t)|\right)\right\}
\end{aligned}
\qquad (43)
$$

From the above equations, it is clear that entanglement between the pair of subsystems 1 and 2 can only arise when the initial state of the system includes the state $|\chi_0\rangle=|-,-,m_1,m_2\rangle$. We also note that the probability amplitude s_0 will remain constant over time. Furthermore, because the form of the interference terms is the difference between the products of the probability amplitude measures, the behavior will be discontinuous. This discontinuous behavior leads to the sudden death of entanglement (ESD) over a period of time. There is also a competition for the creation of atom–atom entanglement (C_{AB}) between the combinations of states $(|\chi_1\rangle,|\chi_4\rangle)$ and $(|\chi_2\rangle,|\chi_3\rangle)$. In fact, the entanglement generated in the pair of atoms and fields, which is involved with the initial states $|\chi_1\rangle,|\chi_2\rangle$, decreases in the presence of states $|\chi_2\rangle,|\chi_3\rangle$, and vice versa.

D. *Sum of Squares of Non-local Intersections*

In this section, we show that the sum of squares of the non-local intersection terms satisfies inequality (11). The reduced density matrix for the atom-atom system in the bases of $|+,+\rangle,|+,-\rangle,|-,+\rangle,|-,-\rangle$ states is as follows:

$$
\rho_{AB}=\begin{bmatrix}
|s_1|^2 & 0 & 0 & s_1s_0^*\\
0 & |s_2|^2 & 0 & 0\\
0 & 0 & |s_3|^2 & 0\\
s_1^*s_0 & 0 & 0 & |s_4|^2+|s_0|^2
\end{bmatrix} \qquad (44)
$$

In this case, the entanglement between the atoms is obtained from the following relation:

$$
C_{AB}(t)=2\left(|s_1(t)||s_0(t)|-|s_2(t)||s_3(t)|\right) \qquad (45)
$$

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In this case, for a period of time, even when the non-diagonal coherent elements $s_1 s_0^*$, which are the source of entanglement, are non-zero, we observe the sudden death of entanglement. In fact, with the decrease in the sum of non-local crossings (SSPC), we will witness the sudden death of entanglement. Indeed, for the Φ state introduced in Eq. (8), there are regions where all the entanglements C_{AB}, $C_{B_1B_2}$, C_{B_1B}, and C_{AB_1} are zero, which means that there will be no non-local entanglement between systems 1 and 2. This effect has been investigated in reference [38] and is plotted in Fig. 5(a) for the case of non-deformed multi-photon fields $m_1 = m_2 = 30$ and $f_1(m_1) = f_2(m_2) = 1$. It is shown for the $\theta = \pi/6$ state. In this case, the entanglement between atoms and fields will be specific to the four states C_{AB}, $C_{B_1B_2}$, C_{B_1B}, and C_{AB_1}. From Eq. (27), it can be determined that when we have $|s_1| = |s_2| = |s_3| = |s_4| = 1/2$, for non-zero values of $s_0 = \sin(\theta)$, all entanglements between the pairs of systems 2 and 1 will be zero. For the initial state:

$$\frac{1}{2}\cos(\theta)\left[|+,+,m_1,m_2\rangle + |+,-,m_1,2m_2\rangle + |-,+,2m_1,m_2\rangle\right] + \sin(\theta)|-,-,m_1,m_2\rangle \quad (46)$$

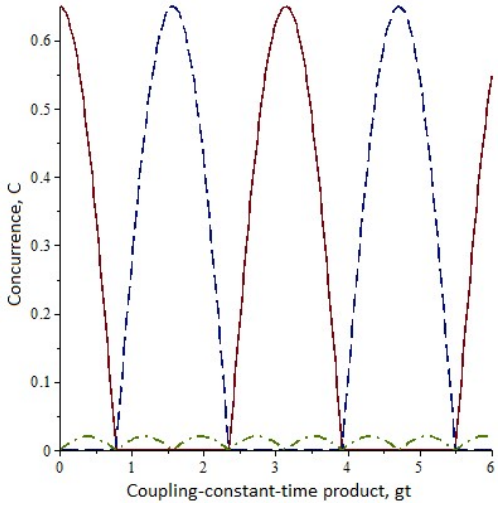
and under the condition $\tan(\theta) < \frac{1}{2}$, all entanglements between systems 2 and 1 will become zero at times $t = n\pi/4\Omega$, $(n = 1, 2, 3, ...)$. In the case of deformed two-mode fields $\left[m_1 = m_2 = 30 \text{ and } f_1(m_1) = f_2(m_2) = \sqrt{m}\right]$, as shown in Fig. 5(b), the entanglement behavior becomes more oscillatory and the time interval during which entanglement becomes zero decreases.

E. *Controlling ESD using Squeezing*

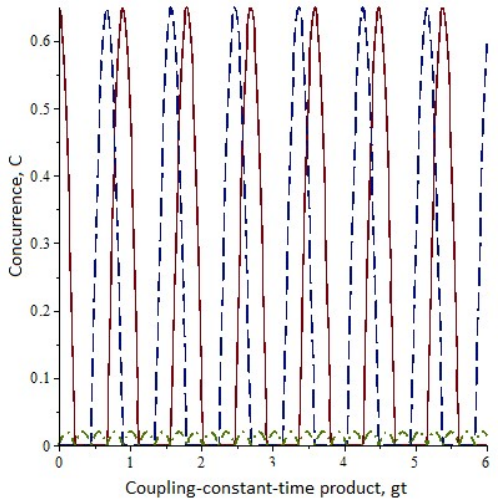
Sudden death of entanglement by Yu and colleagues in reference [34] for the case where the initial state of the system is as follows:

$$|\chi\rangle = \cos(\theta)|+,+,m_1,m_2\rangle + \sin(\theta)|-,-,m_1,m_2\rangle \quad (47)$$

has been investigated, with identical coupling constants and zero squeezing.



(a)



(b)

Fig. 5. (a) Plot of the squared time evolution of entanglement for the case $\theta = \pi/6$, $\Delta' = 0$, and $g_1 = g_2' = g$ C_{AB}^2 (solid line), $C_{B_1B_2}^2$ (dashed line), $C_{AB_2}^2 = C_{B_1B}^2$ (dash-dot line). Here we have assumed $m_1 = 30$, $m_2 = 30$, and $f_1(m) = f_2(m) = 1$. (b) Plot of the squared time evolution of entanglement for the case $\theta = \pi/6$, $\Delta' = 0$, and $g_1' = g_2' = g$, C_{AB}^2 (solid line), $C_{B_1B_2}^2$ (dashed line), $C_{AB_2}^2 = C_{B_1B}^2$ (dash-dot line). Here we have assumed $m_1 = m_2 = 30$ and $f_1(m) = f_2(m) = \sqrt{m}$.

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They showed that for the $\theta \neq \pi/4$ state, entanglement will disappear discontinuously. We have illustrated this for the $\theta = \pi/12$ state in Fig. 6. Furthermore, we observe that when $\tan(\theta)$ is not equal to one, the time duration during which entanglement disappears increases. We also observe that squeezing can prevent the sudden death of entanglement, and this will lead to the stability of entanglement. Fig. 6 also shows that squeezing prevents the complete transfer of entanglement from the atoms to the cavity fields. However, for some initial conditions, squeezing will cause the sudden death of entanglement and lead to instability, while for sufficiently large values of squeezing, it reaches a stable state. To further investigate the effects of the presence of squeezing on the time evolution of entanglement, we consider two different initial conditions. In one of these initial conditions, zero squeezing causes the sudden death of entanglement (ESD), and in the other, $\Delta = 0$ causes the stability of entanglement. In the first case, we consider the initial state function as a symmetric combination of $(|\chi_1\rangle, |\chi_4\rangle, |\chi_0\rangle)$ states:

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|+,+,m_1,m_2\rangle + |-,-,2m_1,2m_2\rangle + |-,-,m_1,m_2\rangle\right)\right) \quad (48)$$

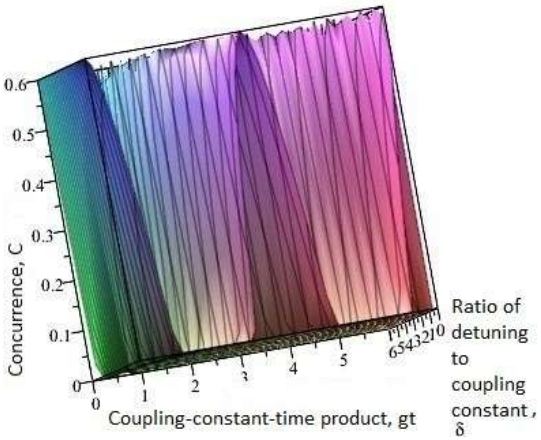


Fig. 6. Three-dimensional plot of atom-atom concurrence $C_{AB_1}(t)$ as a function of time and $\delta = |\Delta'/g'|$ for the initial condition $|\Phi\rangle = \cos(\theta)|+,+,m_1,m_2\rangle + \sin(\theta)|-,-,m_1,m_2\rangle$. Here $\theta = \pi/12$, $\Delta' \neq 0$, and $g_1' = g_2' = g'$.

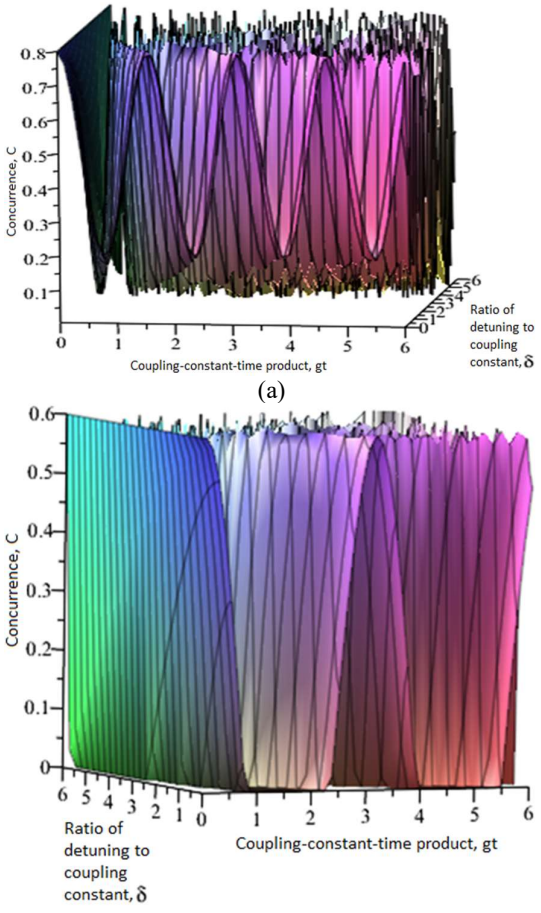


Fig. 7. (a) Three-dimensional plot of atom-atom concurrence (C_{AB}) as a function of time and $\delta = |\Delta'/g'|$ for the asymmetric initial condition, (b) Three-dimensional plot of atom-atom concurrence (C_{AB}) as a function of time and $\delta = |\Delta'/g'|$ for the symmetric initial condition.

In the second case, we consider the initial state function as an asymmetric combination of $(|\chi_1\rangle, |\chi_4\rangle, |\chi_0\rangle)$ states:

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|+,+,m_1,m_2\rangle - |-,-,2m_1,2m_2\rangle + |-,-,m_1,m_2\rangle\right)\right) \quad (49)$$

In Fig. 7(a), the time evolution diagram of entanglement $C_{AB}(t)$ as a function of squeezing for the $\Delta_1 = \Delta_2 = \Delta$ case is plotted for the asymmetric initial condition. We observe that in the asymmetric case, zero squeezing ($\Delta = 0$) leads to the stability of entanglement, while non-zero squeezing causes the sudden death of entanglement (ESD). Whereas for the

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symmetric initial condition (Fig. 7(b)), we observe unstable squeezing and the (ESD) phenomenon in the case of zero squeezing $(\Delta = 0)$, and in the $\Delta \neq 0$ case, entanglement stability occurs.

IV. CONCLUSION

In this research, we investigated the entanglement between two cavities containing a two-level atom and a deformed multi-photon field. The model used in this study is the deformed Jaynes-Cummings model (DJCM). A physical interpretation of the introduced model is that there are two separate cavities with Kerr-like media, each containing a two-level atom and a non-deformed multi-photon field, where each atom interacts locally with its corresponding field. We showed that in the case where subsystems 1 and 2 become indistinguishable, greater stability in entanglement will be observed. We will also demonstrate that the presence of asymmetric cavities and the presence of nonlinearity are factors that can be used to control entanglement. We examined the time evolution diagram of C_{AA_1} for the initial state $|+,+,m_1,m_2\rangle$ and observed that when the excitation is equally shared between the atom and the field, maximum entanglement between the atom and the field occurs. Additionally, when the excitation is concentrated solely on either the atom or the field, the entanglement between the atom and the field disappears. Also, when there is no detuning, the entanglement between the atom and the field will disappear at times $t = n\pi/g\Omega$, $(0,1,2,...)$. Furthermore, we observe that when entanglement exists, the entanglement time of the system increases. In the presence of entanglement, the entanglement of the system will become zero at times $t = 2n\pi/g\Omega$, $(n = 0,1,2,...)$. For the deformed state $m_1 = m_2 = 30$ and $f_1(m_1) = f_2(m_2) = \sqrt{m}$, we observed that the entanglement diagram will become highly oscillatory, which is predictable given the non-Markovian nature of the environment. We also observed that in the deformed state, the time intervals during which entanglement is destroyed will be shorter. We also observed that since the parameters *g* and Ω for the deformed fields are functions of the deformed parameter, it can be concluded that in both cases of zero entanglement and non-zero entanglement, the times at which entanglement becomes zero depend on the number of photons inside the medium. For the case where the initial state of the system is $|\chi_0\rangle = |-,-,m_1,m_2\rangle$, we observed that due to the interference terms appearing as the difference of products of amplitude operators, the behavior will be discontinuous. This discontinuous behavior will cause sudden death of entanglement (ESD) over a period of time. In a case where the initial state of the system is $|\chi\rangle = \cos\theta|+,+,m_1,m_2\rangle + e^{i\alpha}\sin\theta|-,-,m_1,m_2\rangle$, we observe that for $\theta = \pi/6$, all the concurrences C_{AB}, $C_{B_1B_2}$, C_{B_1B}, and C_{AB_1} are zero, which means that there will be no nonlocal entanglement between systems 1 and 2. In this case, the entanglement between atoms and fields will be limited to the four C_{AB}, $C_{B_1B_2}$, C_{B_1B}, and C_{AB_1} states. Furthermore, we showed that in the scenario where the initial state is $|\chi\rangle = \cos(\theta)|+,+,m_1,m_2\rangle + \sin(\theta)|-,-,m_1,m_2\rangle$, $\theta = \pi/12$ present, we observed that entanglement prevents the sudden disappearance of entanglement, and this will lead to the stability of entanglement. Regarding entanglement environments, since de-coherence depends on the number of photons, we conclude that the stability or instability of entanglement depends on the number of photons in the cavity. In fact, by adjusting the number of photons inside the cavity, de-coherence can be controlled, ultimately leading to the stability of entanglement. We also observed that de-coherence prevents the complete transfer of entanglement from atoms to the cavity fields. Furthermore, we observed that in the asymmetric case, zero de-coherence $(\Delta = 0)$ leads to entanglement stability, while non-zero de-coherence causes entanglement sudden death (ESD). In addition, we observed that for symmetric initial conditions, the ESD phenomenon occurs in the zero-detuning case

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$(\Delta = 0)$, while in the $\Delta \neq 0$ state, entanglement stability is observed.

APPENDIX A

Proof of Eq. (19): Due to the similarity of Eqs. (19)-(22), we prove Eq. (19). Using Eqs. (13) and (14), we can write

$$\begin{aligned}&i\hbar\dot{s}_1(t)\left|+,+,m_1,m_2\right\rangle + i\hbar\dot{s}_2(t)\left|+,-,m_1,2m_2\right\rangle + \\ &i\hbar\dot{s}_3(t)\left|-,+,2m_1,m_2\right\rangle + i\hbar\dot{s}_4(t)\left|-,-,2m_1,2m_2\right\rangle = \\ &\left[\hbar\omega_1\hat{B}_1^{\dagger}\hat{B}_1 + \hbar\omega_2\hat{B}_2^{\dagger}\hat{B}_2 + \hbar\omega_0 s_A^z + \hbar\omega_0 s_B^z + \right. \\ &\left. + \hbar g_1\left((\hat{B}_1^{m_1})^{\dagger}s_A^- + \hat{B}_1^{m_1}s_A^+\right) + \hbar g_2\left((\hat{B}_2^{m_2})^{\dagger}s_B^- + \hat{B}_2^{m_2}s_B^+\right)\right] \\ &\left[\left|+,+,m_1,m_2\right\rangle + \left|+,-,m_1,2m_2\right\rangle + \left|-,+,2m_1,m_2\right\rangle + \right. \\ &\left.\left|-,-,2m_1,2m_2\right\rangle\right]\end{aligned} \tag{A-1}$$

Now we use the relations

$$\begin{aligned}&\hat{s}_z\left|+\right\rangle = \frac{1}{2}\left|+\right\rangle, \hat{s}_z\left|-\right\rangle = -\frac{1}{2}\left|-\right\rangle, \hat{s}^+\left|+\right\rangle = \\ &0, \hat{s}^-\left|-\right\rangle = 0, \hat{s}^+\left|-\right\rangle = \left|+\right\rangle, \hat{s}^-\left|+\right\rangle = \left|-\right\rangle\end{aligned} \tag{A-2}$$

$$\hat{B}_i^{\dagger}\hat{B}_i\left|n_i\right\rangle = \hat{m}_i f_i^2(\hat{m}_i)\left|m_i\right\rangle = m_i f_i^2(m_i)\left|m_i\right\rangle \tag{A-3}$$

$$\begin{aligned}&\hat{B}_j^{\dagger q}\left|n_j\right\rangle = f_j^q(n_j+q)\sqrt{\frac{(n_j+q)!}{n_j!}}\left|n_j+q\right\rangle \\ &\hat{B}_i^{p}\left|n_i\right\rangle = f_i^p(n_i)\sqrt{\frac{n_i!}{(n_i-p)!}}\left|n_i-p\right\rangle\end{aligned} \tag{A-4}$$

By equating the coefficients of $\left|\uparrow,\uparrow,m_1,m_2\right\rangle$ on both sides of Eq. (A-1), we arrive at the following relation:

$$\begin{aligned}&\left[\omega_1 n_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right] - i\dot{s}_1 = \\ &(\Delta_1' + \Delta_2')s_1 + (g_2' s_2 + g_1' s_3)\end{aligned} \tag{A-5}$$

Here Δ_j', g_1', and g_2' are the given relations in Eqs. (23)-(25). By multiplying both sides of the above relation by the term $\exp\left\{-i\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right]t\right\}$, we obtain:

$$\begin{aligned}&\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right]e^{-i\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right]t} \\ &-i\dot{s}_1 e^{-i\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right]t} = (\Delta_1' + \Delta_2')\times \\ &\times s_1 e^{-i\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right]t} + \left[g_2' s_2 e^{-i\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2 m_2\right]t} + \right. \\ &\left. + g_1' s_3 e^{-i\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right]t}\right]\end{aligned} \tag{A-6}$$

Defining $\tilde{s}_i = e^{-i\left[\omega_1 m_1 f_1^2(m_1) + \omega_2 m_2 f_2^2(m_2)\right]t} s_i$, we arrive at Eq. (19).

APPENDIX B

Proof of Eq. (31): Eqs. (19)-(22), assuming:

$$\Delta_1' = \Delta_2' = \Delta', g_1' = g_2' = g \tag{B-1}$$

They turn out as follows:

$$\dot{\tilde{s}}_1 = -i(\Delta + \Delta)\tilde{s}_1 - i(g\tilde{s}_2 + g\tilde{s}_3) = -i(2\Delta)\tilde{s}_1 - ig(\tilde{s}_2 + \tilde{s}_3) \tag{B-2}$$

$$\dot{\tilde{s}}_2 = -i(\Delta - \Delta)\tilde{s}_2 - i(g\tilde{s}_1 + g\tilde{s}_4) = -ig(\tilde{s}_1 + \tilde{s}_4) \tag{B-3}$$

$$\dot{\tilde{s}}_3 = i(\Delta - \Delta)\tilde{s}_3 - i(g\tilde{s}_1 + g\tilde{s}_4) = -ig(\tilde{s}_1 + \tilde{s}_4) \tag{B-4}$$

$$\dot{\tilde{s}}_4 = i(\Delta + \Delta)\tilde{s}_4 - i(g\tilde{s}_2 + g\tilde{s}_3) = i(2\Delta)\tilde{s}_4 - -ig(\tilde{s}_2 + \tilde{s}_3) \tag{B-5}$$

By defining new variables:

$$\begin{aligned}s^{(1)} &= \tilde{s}_1 + \tilde{s}_4 \\ s^{(2)} &= \tilde{s}_1 - \tilde{s}_4 \\ s^{(3)} &= \tilde{s}_2 + \tilde{s}_3 \\ s^{(4)} &= \tilde{s}_2 - \tilde{s}_3\end{aligned} \tag{B-6}$$

For $\dot{s}^{(1)}$, we can write:

$$\begin{aligned}&\dot{s}^{(1)} = \dot{\tilde{s}}_1 + \dot{\tilde{s}}_4 = -i(2\Delta)\tilde{s}_1 - ig(\tilde{s}_2 + \tilde{s}_3) + i(2\Delta)\tilde{s}_4 \\ &-ig(\tilde{s}_2 + \tilde{s}_3) = -i(2\Delta)(\tilde{s}_1 - \tilde{s}_4) - 2ig(\tilde{s}_2 + \tilde{s}_3) = \\ &-i(2\Delta)s^{(2)} - 2igs^{(3)}\end{aligned} \tag{B-7}$$

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For $\dot{d}^{(2)}$, we can write:

$$\dot{s}^{(2)} = \dot{\tilde{s}}_1 - \dot{\tilde{s}}_4 = -i(2\Delta)\tilde{s}_1 - ig(\tilde{s}_2 + \tilde{s}_3) - i(2\Delta)\tilde{s}_4 \\ -ig(\tilde{s}_2 + \tilde{s}_3) = -i(2\Delta)(\tilde{s}_1 + \tilde{s}_4) - 2ig(\tilde{s}_2 + \tilde{s}_3) + \\ ig(\tilde{s}_2 + \tilde{s}_3) = -i(2\Delta)s^{(1)} \tag{B-8}$$

For $\dot{s}^{(3)}$, we can write:

$$\dot{s}^{(3)} = \dot{\tilde{s}}_2 + \dot{\tilde{s}}_3 = -ig(\tilde{s}_1 + \tilde{s}_4) - ig(\tilde{s}_1 + \tilde{s}_4) = \\ -2ig(\tilde{s}_1 + \tilde{s}_4) = -2igs^{(1)} \tag{B-9}$$

For $\dot{s}^{(4)}$, we can write:

$$\dot{s}^{(4)} = \dot{\tilde{s}}_2 - \dot{\tilde{s}}_3 = -ig(\tilde{s}_1 + \tilde{s}_4) + ig(\tilde{s}_1 + \tilde{s}_4) = 0 \tag{B-10}$$

Then $s^{(4)}(t) = s^{(4)}(0)$ is constant.

Thus, the system of differential equation is obtained as follows:

$$\dot{s}^{(1)} = -2i\Delta s^{(2)} - 2igs^{(3)} \\ \dot{s}^{(2)} = -2i\Delta s^{(1)} \\ \dot{s}^{(3)} = -2gs^{(1)} \\ \dot{s}^{(4)} = 0 \tag{B-11}$$

By differentiating from the first equation and substituting from the second and third equations we obtain:

$$\ddot{s}^{(1)} = -2i\Delta\dot{s}^{(2)} - 2ig\dot{s}^{(3)} = -2i\Delta\left(-2i\Delta s^{(1)}\right) - \\ 2ig\left(-2igs^{(1)}\right) = \left(4\Delta^2 + 4g^2\right)\left(-s^{(1)}\right) = -4\left(g^2 + \Delta^2\right)s^{(1)} \tag{B-12}$$

Then we can write:

$$\ddot{s}^{(1)} + \Omega_g^2 s^{(1)} = 0 \tag{B-13}$$

Here we have defined:

$$\Omega_g = 2\sqrt{g^2 + \Delta^2} = 2g\sqrt{1 + \delta^2} \tag{B-14}$$

By solving Eq. (B-13) we can write:

$$s^{(1)}(t) = A\cos\left(\Omega_g t\right) + B\sin\left(\Omega_g t\right) \tag{B-15}$$

Using the initial conditions for the coefficients, we obtain:

$$A = s^{(1)}(0), B = -\frac{2i}{\Omega}\left[s^{(3)}(0) + \delta s^{(2)}(0)\right] \tag{B-4}$$

Putting Eq. (B-16) into Eq. (B-15) we obtain Eq. (31).

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