Refraction at the Interface of a Lossy Metamaterial

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Abstract— **The refraction phenomenon at the interface of an ordinary material and a lossy metamaterial has been investigated. For oblique incidence on the lossy metamaterial, the planes of constant amplitude of the refracted wave are parallel to the interface and the plane of constant phases make a real angle with the interface (real refraction angle). The real refraction angle and hence, the real refraction index corresponds to the real refraction angle which satisfies the real version of Snell's law are negative in two different regimes. In one regime, the metamaterial is double-negative, while in the other one it is single-negative. Moreover, we show that the plane wave solution for the refracted wave is causal in both double-negative and single-negative regimes.**

KEYWORDS: Refraction, lossy metamaterial, single-negative, double-negative.

I. INTRODUCTION

In classical electrodynamics, the electromagnetic response of a homogeneous medium is usually characterized by two fundamental quantities, the permittivity ε and the permeability μ. The refractive index n which first arises in the context of the wave equation is defined by $n = \sqrt{\varepsilon\mu}$. If we do not take losses into account and treat ε and μ as real numbers, according to Maxwell's equations, electromagnetic waves can propagate through a material only if the index of refraction n, is real. In a lossless double positive (DPS) medium with ε and μ both positive the index of refraction n is real. In a single-negative (SNG) medium where one of ε and μ is negative but the other is positive, the index of refraction is imaginary and electromagnetic waves cannot propagate. Metals are such kind of media. In fact, the electromagnetic response of metals in the visible and near-ultraviolet regions is dominated by the negative epsilon concept [1, 2]. Although all our everyday transparent materials are DPS, from the theoretical point of view, in a double negative (DNG) medium with ε and μ both negative, the electromagnetic waves can also propagate through it. The topic of DNG metamaterials, as specially engineered materials has received ever increasing attention in recent years. This was first pointed out by Veselago when no material with simultaneously negative ε and μ was known [3]. Moreover, if such media exist, the propagation of waves through them should give rise to a variety of unusual properties of electromagnetic waves. For example, the cross product of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ for a plane wave in regular media gives the direction of both propagation and energy flow, and the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, and the wave vector $\mathbf{k}$ form a right-handed triplet of vectors. In contrast, in a medium with ε and μ both negative, $\mathbf{E}\times\mathbf{H}$ for a plane wave still gives the direction of energy flow, but the wave itself that is, the phase velocity propagates in the opposite direction, i.e., wave vector $\mathbf{k}$ lies in the opposite direction of $\mathbf{E}\times\mathbf{H}$ for propagating waves. In this medium, electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, and wave vector $\mathbf{k}$ form a left-handed triplet of vectors. Such a medium is therefore termed left-handed medium [4]. Left-handed medium or metamaterial was originally proposed for the microwave domain [5, 6] and then introduced into terahertz frequencies [7]. Simulations and experiments have demonstrated the properties of metamaterials from radio frequencies to THz [2]. The permittivity and permeability of the metamaterials are always frequency dependent, i.e., the metamaterials are always dispersive materials. In order to satisfy the causality conditions dispersive metamaterials

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should be lossy [8, 9]. Materials that exhibit losses have a complex valued refractive index $n = n' + i\,n''$. The real part of complex refraction (n') is called refraction index of the lossy materials. For a plane wave propagating along the z-axis in a uniform lossy medium with time dependence $e^{-i\omega t}$ the imaginary part of complex refraction (n'') which shows the effect of lossed, should be a positive number in order to meet the energy conservation law when z tends to positive infinity. Nevertheless, when considering the propagation of electromagnetic waves, using a complex refractive index is generally inconvenient. On the other hand, electromagnetic waves propagating in such a material usually enter the material through a boundary between it and other medium.

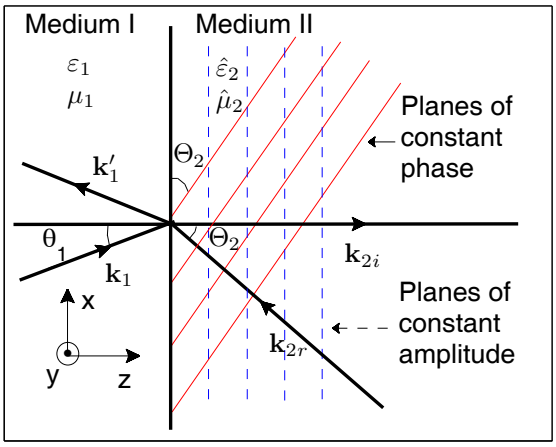

Fig. 1 (Color online) Refraction into a lossy DNG material. The planes of constant phase are perpendicular to the propagation direction and make a negative angle θ_2 with the interface, but the planes of constant amplitude are parallel to the interface.

Here, we want to treat the propagation of electromagnetic waves incident from an ordinary medium on a plane interface of a lossy metamaterial. We show that the transmitted wave is refracted at a negative angle and it is attenuated in the direction normal to the interface. Due to the Snell's low, negative refraction angle is equivalent to a real and negative refraction index. We demonstrate that the real refraction index of lossy metamaterial can be negative at two different regimes. In one regime, the lossy metamaterial is double negative and in the other one it is single negative.

II. Theoretical Model

We choose the geometry shown in Fig. 1, in which a plane wave is incident obliquely from an ordinary medium with refractive index $n_1 = \sqrt{\varepsilon_1 \mu_1}$ on a plane interface of a lossy metamaterial. The permittivity ε_1 and permeability μ_1 of the ordinary medium are considered to be real positive quantities. The complex permittivity and permeability of the metamaterial are considered as $\hat{\varepsilon}_2 = \varepsilon'_2 + i\varepsilon''_2$ and $\hat{\mu}_2 = \mu'_2 + i\mu''_2$, respectively. Here, we assume that the imaginary parts of $\hat{\varepsilon}_2$ and $\hat{\mu}_2$ are positive without any limitation on the sign of their real parts. We choose the z axis normal to the interface, the x axis in the plane of the figure, and the y axis out of the plane of the figure. Here, $\mathbf{k_1}, \mathbf{k'_1}$ are the wave vectors of incident and reflected waves, respectively, and $\hat{\mathbf{k}}_\mathbf{2}$ is the complex wave vector of transmitted wave. The electric (magnetic) fields of incident, reflected, and transmitted TE-polarized (TM-polarize) waves are considered as

$$\begin{aligned} \mathbf{F}_i &= \mathbf{e}_y F_i e^{i(\mathbf{k}_1 . r - \omega t)}, \\ \mathbf{F}_r &= \mathbf{e}_y F_r e^{i(\mathbf{k}'_1 . r - \omega t)}, \\ \mathbf{F}_t &= \mathbf{e}_y F_t e^{i(\hat{\mathbf{k}}_2 . r - \omega t)}. \end{aligned} \tag{1}$$

The wave vectors $\mathbf{k_1}$, $\mathbf{k'_1}$ are written as

$$\begin{aligned} \mathbf{k}_1 &= k_{1x}\mathbf{e}_x + k_{1z}\mathbf{e}_z, \\ \mathbf{k}'_1 &= k_{1x}\mathbf{e}_x - k_{1z}\mathbf{e}_z. \end{aligned} \tag{2}$$

Here, $k_{1x} = \dfrac{\omega}{c} n_1 \sin\theta_1$, which remains constant throughout the media, is the tangential component of all the wave vectors. The complex wave vector of the transmitted wave is described by two equivalent expressions

$$\hat{\mathbf{k}}_2 = \mathbf{k}_{2r} + i\mathbf{k}_{2i} = \hat{k}_2 \sin\hat{\theta}_2 \mathbf{e}_x + \hat{k}_2 \cos\hat{\theta}_2 \mathbf{e}_z \tag{3}$$

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Here, $\hat{k}_2$ is a scalar which is defined as $\hat{k}_2 = \sqrt{\hat{\mathbf{k}}_2.\hat{\mathbf{k}}_2}$ and used in the dispersion relation $\hat{k}_2 = \hat{\varepsilon}_2\hat{\mu}_2\omega^2/c^2$ and $\hat{\theta}_2$ is the complex refraction angle. Inserting the complex wave vector $\hat{\mathbf{k}}_2$ from Eq. (3) into the Eq. (1), yields

$$\mathbf{F}_t = \mathbf{e}_y F_t e^{-\mathbf{k}_{2i}.r} e^{(i\mathbf{k}_{2r}.r-\omega t)}, \tag{4}$$

This eqation reveals that $\mathbf{F}_t$ is a plane wave which propagates in the direction $\mathbf{k}_{2r}$ and instead having a constant amplitude, depending on the sign of $\mathbf{k}_{2i}$, it decreases or increases in the direction $\mathbf{k}_{2i}$. The surfaces of constant phase are planes perpendicular to the propagation direction $\mathbf{k}_{2r}$. There are also surfaces of constant amplitude that are planes perpendicular to $\mathbf{k}_{2i}$ (see Fig. 1). In order to determine the directions and the values of the real and the imaginary parts of the complex wave vector $\hat{\mathbf{k}}_2$, we use the boundary conditions at the interface of two media. According to the boundary conditions, not only the reflected and transmitted waves must have the same frequency as the incident wave, but also their phases must match everywhere on the boundary. Hence,

$$\hat{\mathbf{k}}_2 \times \mathbf{n} = \mathbf{k}_1' \times \mathbf{n} = \mathbf{k}_1 \times \mathbf{n} \tag{5}$$

Here, $\mathbf{n} = \mathbf{e}_z$ is the unit vector perpendicular to the interface. This equation means that $\hat{\mathbf{k}}_2 \times \mathbf{n}$ is real, so

$$\mathbf{k}_{2i} \times \mathbf{n} = 0, \qquad \mathbf{k}_{2r} \times \mathbf{n} = \mathbf{k}_1 \times \mathbf{n} \tag{6}$$

i.e. $\mathbf{k}_{2i} = k_{2i}\mathbf{e}_z$ is parallel to the z axis, and

$$k_{2r} \sin\theta_2 = k_1 \sin\theta_1, \tag{7}$$

where θ_2 is the real angle between $\mathbf{k}_{2r}$ and $\mathbf{k}_{2i}$ (i.e. the angle between the planes of constant phase and the interface). By rewriting the wave vector $\hat{\mathbf{k}}_2$ as

$$\begin{aligned}\hat{\mathbf{k}}_2 &= k_{2r}\sin\theta_2\mathbf{e}_x + k_{2r}\cos\theta_2\mathbf{e}_z + ik_{2i}\mathbf{e}_z \\ &= k_1\sin\theta_1\mathbf{e}_x + (k_{2r}\cos\theta_2 + ik_{2i})\mathbf{e}_z,\end{aligned} \tag{8}$$

and using Eq. (3), we see that

$$k_1\sin\theta_1 = \hat{k}_2\sin\hat{\theta}_2,\ k_{2r}\cos\theta_2 + ik_{2i} = \hat{k}_2\cos\hat{\theta}_2 \tag{9}$$

Using these equations, together with the abbreviation $\hat{k}_2\cos\hat{\theta}_2 = \frac{\omega}{c}(p+iq)$, gives

$$k_{2i} = \frac{\omega}{c}q, \qquad k_{2r}\cos\theta_2 = \frac{\omega}{c}p. \tag{10}$$

Here, $k_{2r} = \frac{\omega}{c}N_2$ and

$$N_2 = \pm\sqrt{p^2 + \varepsilon_1\mu_1\sin^2\theta_1}. \tag{11}$$

is the real refractive index of the medium II which gives the phase velocity as c/N_2, and satisfies in a real version of Snell's law,

$$N_2\sin\theta_2 = n_1\sin\theta_1, \qquad N_2\cos\theta_2 = p. \tag{12}$$

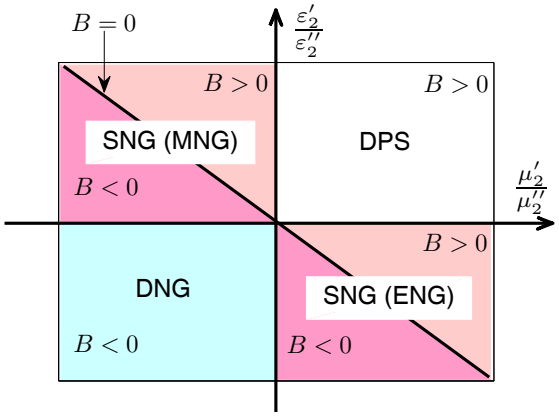

Fig. 2 (Color online) The parameter $B=(\varepsilon_2'/\varepsilon_2''+\mu_2'/\mu_2'')$ in the plane of $\varepsilon_2'/\varepsilon_2'' - \mu_2'/\mu_2''$.

The values of p and q are satisfying in the following relations which can be obtained immediately from Eq. (9)

$$\begin{aligned}p^2 - q^2 &= (\varepsilon_2''\mu_2'')\left(\frac{\varepsilon_2'\mu_2'}{\varepsilon_2''\mu_2''} - 1 - \frac{n_1^2\sin^2\theta_1}{\varepsilon_2''\mu_2''}\right), \\ 2pq &= (\varepsilon_2''\mu_2'')\left(\frac{\varepsilon_2'}{\varepsilon_2''} + \frac{\mu_2'}{\mu_2''}\right).\end{aligned} \tag{13}$$

Solving Eq. (13) gives

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$$\frac{q}{\sqrt{\varepsilon_2''\mu_2''}}=\pm\sqrt{\frac{1}{2}\left(-A+\sqrt{A^2+B^2}\right)},\quad \frac{p}{\sqrt{\varepsilon_2''\mu_2''}}=\frac{B}{2q} \tag{14}$$

where $A=\left(\frac{\varepsilon_2'\mu_2'}{\varepsilon_2''\mu_2''}-1-\frac{n_1^2\sin^2\theta_1}{1\varepsilon_2''\mu_2''}\right)$ and $B=\left(\frac{\varepsilon_2'}{\varepsilon_2''}+\frac{\mu_2'}{\mu_2''}\right)$.

III. Results and Discussion

As one can see, the values of p, q, N_2 and θ_2 depend on the incidence angle θ_1. Since, q shows the effect of loss, exponential decay of the wave in lossy materials requires that $q>0$. So, we choose positive sign in Eq. (14). Then, the sign of p, and hence the sign of N_2 are determined by the sign of parameter B. To illustrate this, we plotted the parameter B, we plat it in the plane of $\frac{\varepsilon_2'}{\varepsilon_2''}-\frac{\mu_2'}{\mu_2''}$ (see Fig. 2). As one can see this parameter is positive at the DPS regions of the metamaterial and it is negative at the DNG regions. While, it can be positive or negative at the SNG regions depending on the values of $\frac{\varepsilon_2'}{\varepsilon_2''}$ and $\frac{\mu_2'}{\mu_2''}$. By using the right sign of B, we can show the effect of losses on the refraction of light at the interface of a lossy metamaterial. To do this, the real refractive index N_2 and the real refraction angle θ_2 are plotted as functions of $\frac{\varepsilon_2'}{\varepsilon_2''}$ and $\frac{\mu_2'}{\mu_2''}$ in Fig. 3.

As stated earlier, the real refractive index N_2 is negative in the regions for which $B<0$ (i.e. in the DNG region and the half of SNG regions). Nevertheless, the right sign of N_2 in lossy metamaterials ($\varepsilon_2''>0$ and $\mu_2''>0$) must obey the causality condition. A causal solution has to be so that the waves travel from points of source to the points of observation, no matter how the sign of refractive index changes in the media. Violating the causality condition would result in a negative energy density which cannot be accepted in physics [10]. In order to show that the values of N_2 given by Eqs. 10 and 11 are causal; we use the Poynting vector of the transmitted wave. The z component of average Poynting vector of the transmitted wave at the angular frequency ω can be given as:

$$\mathbf{S}_t.\mathbf{e}_z=\frac{\alpha''\mid F_t\mid^2}{2\left(\alpha'^2+\alpha''^2\right)}\left(\frac{\alpha'}{\alpha''}p+q\right)e^{-2\omega qz/c} \tag{15}$$

with $\frac{n_1^2\sin^2\theta_1}{\varepsilon_2''\mu_2''}=1$.

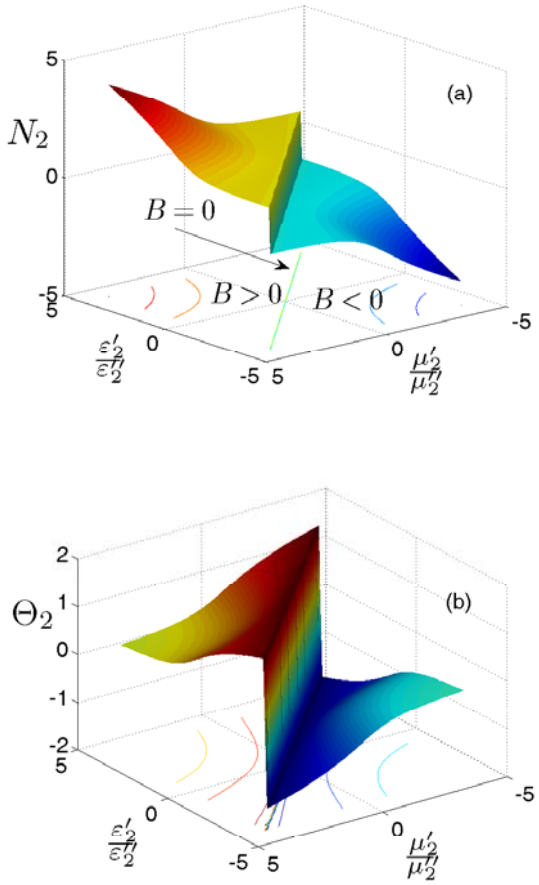

Fig. 3 (Color online) a) Real refractive index N_2 and b) real refraction angle θ_2 as functions of $\varepsilon_2'/\varepsilon_2''$ and μ_2'/μ_2'' for $n_1^2\sin^2\theta_1/\varepsilon_2''\mu_2''=1$.

Here, α' and α'' show the real and the imaginary parts of $\hat{\mu}_2$ ($\hat{\varepsilon}_2$) for TE-polarized (TM-polarized) waves, respectively. Since all parameters in Eq. (9) are positive, one

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concludes that the quantity $\frac{\alpha'}{\alpha''}p+q$ has to be positive to maintain the Poynting vector in the causal direction, i.e.

$$\frac{\alpha'}{\alpha''}p+q>0 \qquad (16)$$

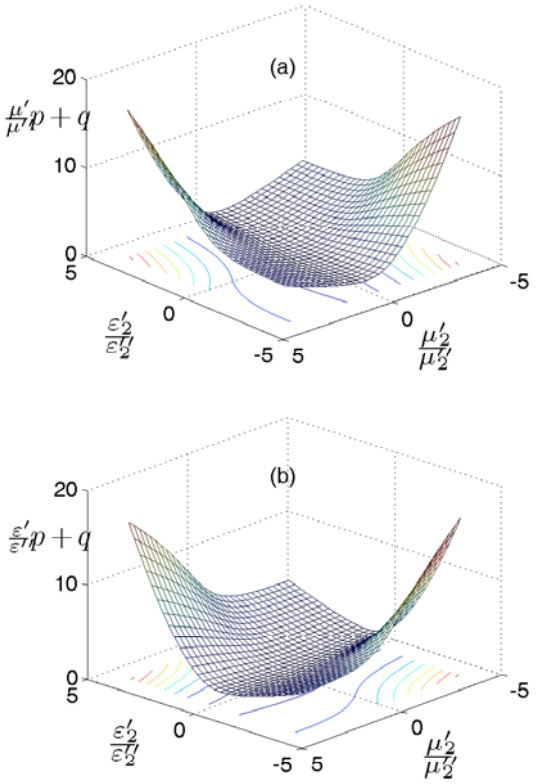

Fig. 4 The quantity $\frac{\alpha'}{\alpha''}p+q$ as functions of $\varepsilon_2'/\varepsilon_2''$ and μ_2'/μ_2'' for (a) TE and (b) TM polarizations, respectively.

In Fig. 4.a-b the quantity $\alpha' p/\alpha''+q$ are plotted as functions of $\varepsilon_2'/\varepsilon_2''$ and μ_2'/μ_2'' for both TE and TM polarizations, respectively. Since $\alpha' p/\alpha''+q$ is positive everywhere in the plane of $\varepsilon_2'/\varepsilon_2''-\mu_2'/\mu_2''$, we conclude that the solutions correspond to the Fig. 3 are causal.

To illustrate the convenience of the above formulism, we apply them to the refraction of light at the interface of a dispersive lossy metamaterial described by the following equations

$$\hat{\varepsilon}_2=1-\omega_{ep}^2/\left(\omega^2+i\gamma_e\omega\right) \qquad (17)$$

$$\hat{\mu}_2=1-F\omega^2/\left(\omega^2-\omega_0^2+i\gamma_m\omega\right) \qquad (18)$$

where ω_{ep} is the plasma frequency, γ_e represents the dissipation of plasmon's energy, F is a geometrical factor, ω_0 is the resonance frequency, and γ_m is the resistive loss. Note that these kinds of dispersion may be realized in special microstrips [11-12]. Figure 5 represents the DNG, the DPS and the SNG frequency ranges of this lossy metamaterial for $\omega_{ep}=10\ \ GHz$, $\omega_0=4\ \ GHz$, $F=0.56$ and $\gamma_e=\gamma_m=0.01\omega$.

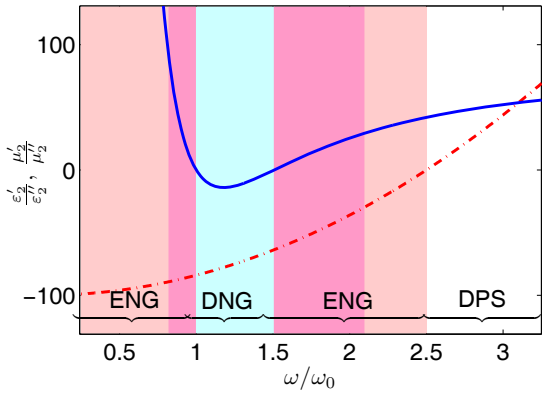

Fig. 5 (Color online) The plot of $\varepsilon_2'/\varepsilon_2''$ and μ_2'/μ_2'' as a function of normalized frequency ω/ω_0 for a lossy metamaterial described by Eqs. (17) and (18) with and $\omega_{ep}=10GHz$, $\omega_0=4GHz$, $F=0.56$ and $\gamma_e=\gamma_m=0.01\omega$

For a plane wave obliquely incident from the vacuum ($n_1=1$), the real refractive index N_2 and the real refractive angle θ_2 of such metamaterial, are shown as functions of normalized frequency ω/ω_0 for three different incidence angles $\theta_1=0^\circ$ (solid lines), $\theta_1=30^\circ$ (dash lines) and $\theta_1=60^\circ$ (dash-dot lines) in Fig. 6. It is seen that the real refractive index N_2, independent from the incidence angles θ_1, is negative at the frequency range $0.82<\omega/\omega_0<2.1$. However, the origin of the negative refraction at the ranges $0.82<\omega/\omega_0<1.0$ and $1.5<\omega/\omega_0<2.1$ is quite different from the

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range $1.0 < \omega/\omega_0 < 1.5$. At the ranges $0.82 < \omega/\omega_0 < 1.0$ and $1.5 < \omega/\omega_0 < 2.1$ the medium II is SNG ($\varepsilon_2' < 0$ and $\mu_2' > 0$). While, the medium II is DNG ($\varepsilon_2' < 0$ and $\mu_2' < 0$). at the range of $1.0 < \omega/\omega_0 < 1.5$.

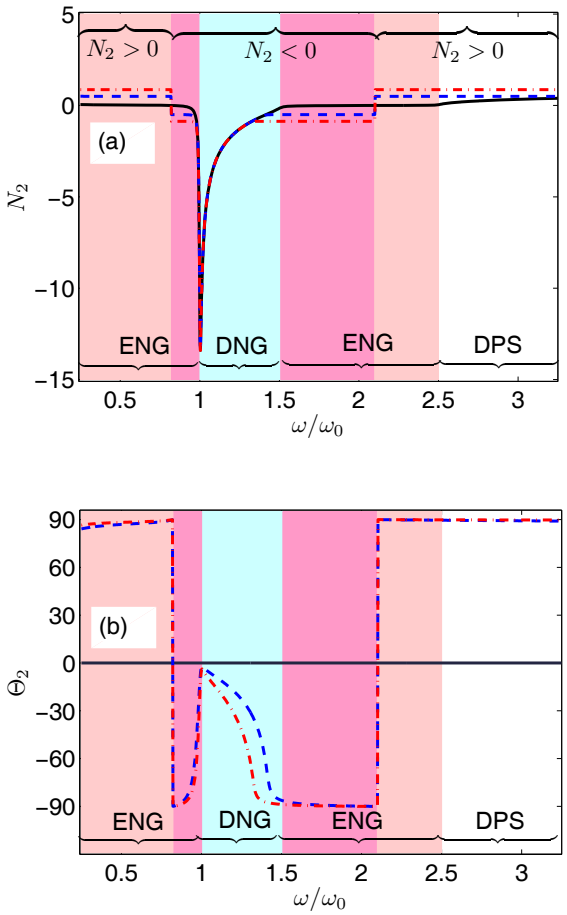

Fig. 6 (Color online) a) real refractive index N_2 and b) real refractive angle θ_2 as functions of normalized frequency ω/ω_0 for three different incidence angles $\theta_1 = 0^\circ$ (solid lines), $\theta_1 = 30^\circ$ (dash lines) and $\theta_1 = 60^\circ$ (dash-dot lines). Other parameters are the same as Fig. 5.

It seems that the design and the fabrication of negative refraction index materials from the SNG materials is much easier than the DNG materials [13, 14]. However, it can be easily shown that the negative refraction with the SNG materials has a larger loss than the DNG one at the same magnitude of the index.

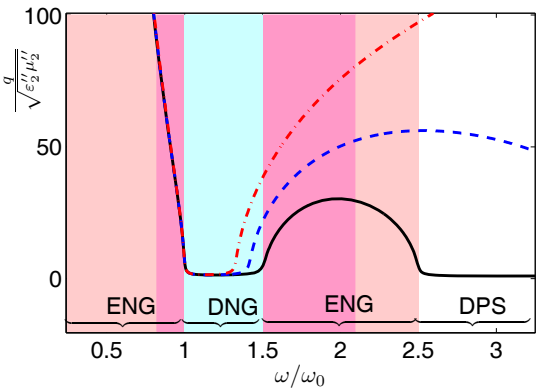

Fig. 7 (Color oline) The same as Fig. 6 for the parameter $q/\sqrt{\varepsilon_2''\mu_2''}$.

To realize this, the parameter $\frac{q}{\sqrt{\varepsilon_2''\mu_2''}}$ is plotted as a function of normalized frequency ω/ω_0 for three different incidence angles $\theta_1 = 0^\circ$ (solid lines), $\theta_1 = 30^\circ$ (dash lines) and $\theta_1 = 60^\circ$ (dash-dot lines) in Fig. 7. From the figure it is seen that the losses at the DNG region are lower than those of the other regions.

IV. Conclusion

In conclusion, for oblique incidence on a lossy metamaterial, the propagation and attenuation of the refracted wave are described. It is stated that the planes of constant amplitude of the refracted wave are parallel to the boundary and the planes of constant phase make a negative real angle with the boundary. It is shown that this real refraction angle and its corresponding real refraction index can be negative at two different regimes. In one regime, the lossy metamaterial is double negative and in the other one it is single negative. It is revealed that the plane wave solutions of the wave equation for the refracted wave are causal in the both regions.

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