Dynamic of Two-Level System in the F-Deformed Jaynes-Cummings Model Beyond the Rotating-Wave Approximation

Mohsen Daeimohammad

Department of Physics, Najafabad Branch, Islamic Azad University, Najafabad, Iran

Corresponding author email: M.Daeimohammad@pco.iaun.ac.ir

Regular paper: Received: Mar. 3, 2021, Revised: Jun. 25, 2021, Accepted: Aug. 25, 2021, Available Online: Aug. 27, 2021, DOI: 10.52547/ijop.15.1.73

ABSTRACT- The aim of this study is to investigate an effective two-level atom coupled to a two-mode f-deformed cavity field with and without the rotating wave approximation. The first section discusses the theoretical model of the interaction between a two-mode cavity-field and an effective two-level atom within the framework of an f-DJCM without the rotating wave approximation. After that, we obtain the reduced density matrix of the atom with and without the rotating-wave approximation. Then, we have investigated the effect of the counter-term on temporal evolution of various non-classical properties of the atom, i.e., atomic population inversion, atomic dipole squeezing and atom-field entanglement. Particularly, we compare the numerical result for three different values of the deformation parameter q (q=1, q=1.1, q=1) with and without the rotating wave approximation.

KEYWORDS: F-Deformed Jaynes-Cummings Model, Rotating Wave Approximation, Counter-Rotating Terms, Virtual-Photon Processes.

I. INTRODUCTION

Many studies have so far discussed the properties of a two-level system (qubit) interacting with a harmonic oscillator [1], [2]. Many of these studies explain the dynamics of the qubit-oscillator system by means of the Rabi.

Hamiltonian [3]:

$$H_{Rabi} = \hbar\omega\hat{a}^2\hat{a} + \hbar\omega_0\hat{\sigma}_z/2 + \\ +\hbar g\left(\hat{a}+\hat{a}^+\right)\hat{\sigma}_x \quad , \tag{1}$$

In most of the experimental studies on cavity QED, the coupling is extremely small (i.e., $\beta = \frac{g}{\omega}$, $\beta \le 10^{-6}$), and the qubit and the oscillator are almost resonant (i.e. $\omega_0 \approx \omega$) [1]. Therefore, according to Jaynes-Cummings (JC) model with the rotating wave approximation (RWA), the rapidly oscillating counter-rotating terms (CRTs) $\hat{a}\hat{\sigma}^-$ and $\hat{a}^+\hat{\sigma}^+$ as virtual photon transitions where $\hat{\sigma}_x = (\hat{\sigma}^+ + \hat{\sigma}^-)$ can be eliminated from the interaction Hamiltonian. However, the RWA cannot be used if the effect of the counter rotating terms is important because of ultra-strong coupling, $|\beta| \ge 0.1$, or because of extremely large detuning, $|\omega - \omega_0| \approx \omega + \omega_0$ [5].

It should be noted that the system has also an interaction with its environment, which results in an incoherent evolution. This phenomenon has been investigated in many studies, considering JC model with the RWA [6]-[11]. The incoherent evolution is modeled by the standard quantum optics master equation (SME). The assumption in SME is that the resulted dissipation mechanism is completely abstract from the qubit-oscillator coupling [12]-[16]. SME predictions are small only when coupling is small. In cases of ultra-strong coupling, SME must not be used since the RWA is not valid.

Today, it is possible to create systems that can operate where JC model with the RWA breaks down [17]-[21]. The most recent investigations on damping in the regime beyond the RWA focus on the effects of increasing the coupling strength while keeping the qubit and oscillator almost resonant with each other [16], [22]-[25].

The generalized JC model with intensity-dependent coupling (IDJC) is the model that has attracted much attention (see, e.g. [26]-[35]). The reason is that this model represents a simple case of a nonlinear interaction corresponding to a more realistic physical situation. The Hamiltonian model in this case is as follows:

$$\hat{H}_{ID} = \hbar g\left(\hat{a}f(\hat{n})\hat{\sigma}^{+} + f(\hat{n})\hat{a}^{+}\hat{\sigma}_{-}\right), \quad (\hat{n} = \hat{a}^{+}\hat{a}) \tag{2}$$

where the function $f(\hat{n})$, which is assumed to be real, describes the intensity dependence coupling of atom-field interaction.

Some researchers are specifically interested in the IDJC model in which $f(\hat{n}) = \sqrt{\hat{n}}$ [26], [27] because of its inherent connection to an $su(1,1)$ JC model [28]. It should also be noted that the IDJC model can potentially provide many variants of the field state possessing interesting quantum statistical features. This means that equation (2) can play the role of a theoretical laboratory for analyzing time evolution of a variety of initial states of the system. Similarly, another theoretical scheme is proposed in [35] that shows the possibility of generating different families of nonlinear coherent states [36], [37] of the radiation field under IDJC model.

On the other hand, the f-deformed Jaynes-Cumming model (f-DJCM) has received much attention because of its connection with quantum algebras [38] which have enabled the researchers to generalize the notion of creation and annihilation operators of the usual quantum oscillator and to introduce a deformed oscillator [39]. In addition, most of the nonlinear generalizations of the JCM are only particular cases of the f-DJCM [40]. The JCM Hamiltonian with an intensity-dependent coupling [41] has been generalized by being related to the quantum $su_q(1,1)$ algebra [42]. This is achieved by using a q-oscillator description.

Various versions of the JCM and their q-deformed extensions have so far been studied and formalized [43]-[45]. The quantum collapse and revival effects of the radiation field in the q-deformed version of the one-photon on-resonant JCM have been investigated [40]. It has been shown [46] that a specific form of the f-DJCM may be realized in a single trapped ion system driven by two laser fields. In [34], the temporal evolution of atomic inversion and quantum fluctuation of the atomic dipole variables have been studied. Coherent states [47] of the radiation field in a lossless coherently pumped micro maser have been proposed [35] through solving a dynamical problem based on a quite general f-DJCM. Coherent states are still extensively investigated; for example, time-dependent q-deformed coherent states, q-coherent and q-cat states [48], and the q-deformed harmonic oscillator with time dependent mass [49] have all been recently studied. In addition, the influence of nonlinear quantum dissipation on the dynamical properties of the one-photon f-DJCM in the large detuning approximation and at zero temperature has been investigated in [50]. In [51] and [52], quantum dynamics of a dissipative deformed harmonic oscillator and a two-mode f-deformed cavity field in a heat bath have been investigated respectively. Additionally, quantum dynamics of a harmonic oscillator in a deformed bath has been investigated in [53]. Also in [54], the influence of the counter-rotating terms on the quantum dynamics of the damped harmonic oscillator in a deformed bath has been investigated. The present study mainly aimed at investigating dynamical properties of the atom by the f-DJCM model with and without the RWA. By the numerical method, we show that even under the condition in which the RWA is considered to be valid, there are significant effects of virtual-photon field on the population inversion, the quantum fluctuation of atomic dipole variables and the atomic linear entropy.

The rest of the paper is organized as follows: first a theoretical background will be given in section II; in section III, the reduced density matrix of the atom will be obtained with and without the rotating-wave approximation. In sections V, we study the effect of counter-rotating terms and parameter deformation on the various atomic properties including population inversion, the quantum fluctuation of atomic dipole variables and the atomic linear entropy. Finally, the study will be summarized and concluded in section VI.

II. The Nondegenerate Two-Photon F-DJCM

In this section, we consider a two-level atom interacting with two modes of an f-deformed cavity field beyond the rotating-wave approximation. The Hamiltonian for this system in the absence of RWA is as follows:

$$\hat{H} = \sum_{i=1.2} \hbar\omega_i \hat{A}_i^+ \hat{A}_i + \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar g\left(\hat{A}_1^+\hat{A}_2^+ + \hat{A}_1\hat{A}_2\right)\left(\hat{\sigma}^+ + \hat{\sigma}^-\right), \quad (3)$$

where ω_1 and ω_2 are frequencies for the two modes of the field, ω_0 is the atomic transition frequency, $\hat{\sigma}_z(|2\rangle\langle 2| - |1\rangle\langle 1|)$ is the atomic inversion operator, $\hat{\sigma}^{\pm}(|2\rangle\langle 1|, |1\rangle\langle 2|)$ are the operators describing the transition between the upper and lower atomic levels, and g is the atom-field coupling constant. For simplicity in the Hamiltonian (3), we ignore the term that describes the intensity-dependent Stark shift of two levels arising due to the transition to an intermediate level. The operators $\hat{A}_i$ and $\hat{A}_i^+$ (i=1,2) are the f-deformed annihilation and creation operators constructed from the conventional bosonic operators $\hat{a}_i$, $\hat{a}_i^+$ $\left([\hat{a}_i, \hat{a}_j^+] = \delta_{ij}\right)$ and number operator, $\hat{n}_i = \hat{a}_i^+ \hat{a}_i$, as follows:

$$\hat{A}_i = \hat{a}_i f_i(\hat{n}_i),\ \hat{A}_i^+ = f_i(\hat{n}_i)\hat{a}_i^+, \quad (4)$$

where $f_i(\hat{n}_i)$ is an arbitrary real function of number operator. The deformed operators $\hat{A}_i$ and $\hat{A}_i^+$ satisfy the f-deformed bosonic oscillator commutation relations:

$$\begin{aligned} &\left[\hat{A}_i, \hat{A}_j^+\right] = \delta_{ij}\left\{\left(\hat{n}_i+1\right) f_i^2\left(\hat{n}_i+1\right) - \hat{n}_i f_i^2\left(\hat{n}_i\right)\right\}, \\ &\left[\hat{A}_i, \hat{n}_j\right] = \delta_{ij}\hat{A}_i \\ &\left[\hat{A}_i^+, \hat{n}_j\right] = -\delta_{ij}\hat{A}_i^+ \end{aligned} \quad (5)$$

In the limiting case, $f_i(\hat{n}_i) = 1$, the Hamiltonian (3) becomes the nondegenerate two-photon non-deformed JC Hamiltonian beyond the RWA [55] and the algebra (5) reduces to the well-known Heisenberg-Weyl algebra generated by $\hat{a}_i$, $\hat{a}_i^+$, and the identity $\hat{I}$. This is very important because it can be directly used in the study of the intensity-dependent atom-field interaction in quantum optics [56] and the study of the quantized motion of a single ion in a harmonic-oscillator potential trap [57]. Accordingly, the Hamiltonian (3) can be written as follows:

$$\hat{H} = \sum_{i=1,2} \hbar\omega_i\hat{n}_i + \sum_{i=1,2}\hbar R_i(\hat{n}_i) + \frac{\hbar}{2}\omega_0\hat{\sigma}_z + \hbar g\begin{bmatrix} f_1(\hat{n}_1) f_2(\hat{n}_2)\hat{a}_1^+\hat{a}_2^+ + \\ +\hat{a}_1\hat{a}_2 f_1(\hat{n}_1) f_2(\hat{n}_2) \end{bmatrix}\left(\hat{\sigma}^+ + \hat{\sigma}^-\right), \quad (6)$$

where $R_i(\hat{n}_i) = \omega_i(f_i^2(\hat{n}_i) - 1)\hat{n}_i$. As it can be seen, Hamiltonian (6) includes forms of the intensity-dependent atom-field coupling, $f_i(\hat{n}_i)$, and the field nonlinearity, $\hat{R}_i(\hat{n}_i)$. In fact, what the Hamiltonian(6) and the Hamiltonian (3) describe is an intensity-dependent two-photon coupling between a single two-level atom and a non-deformed two-mode radiation field when there are two additional nonlinear interactions: $R_1(\hat{n}_1)$ and $R_2(\hat{n}_2)$ beyond the RWA. If $f_i(\hat{n}_i) = \sqrt{1 + k_i(\hat{n}_i - 1)}$, where k_i is a positive constant, the model consists of a single two-level atom interacting via an intensity-dependent two-photon coupling with a two-mode field surrounded by two nonlinear Kerr-like media contained inside a lossless cavity beyond the RWA[34]. Physically, k_1 and k_2 are related to the dispersive part of the third-order nonlinearity of the two Kerr-like media ($\chi_i = k_i \omega_i$).

III. The Reduced Density Matrix of the Atom with and without the Rotating-Wave Approximation

In this section we obtain the reduced density matrix of the atom with and without the rotating-wave approximation. Let us write the Hamiltonian (3) as follows:

$$\hat{H}_T = \hat{H}_0 + \hat{H}'$$
$$\hat{H}_0 = \sum_{i=1,2} \hbar\omega_i \hat{A}_i^+ \hat{A}_i + \frac{\hbar}{2}\omega_0 \hat{\sigma}_z$$
$$\hat{H}' = \hbar g\left(\hat{A}_1^+ \hat{A}_2^+ + \hat{A}_1 \hat{A}_2\right)\left(\hat{\sigma}^+ + \hat{\sigma}^-\right), \tag{7}$$

In the interaction picture we have:

$$\hat{H}_I(t) = e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{H}' e^{\frac{-i\hat{H}_0 t}{\hbar}} = \hbar g\Big[\hat{A}_{I1}^+(t)\hat{A}_{I2}^+(t) + \hat{A}_{I1}(t)\hat{A}_{I2}(t)\Big]\Big[\hat{\sigma}_I^+ + \hat{\sigma}_I^-\Big], \tag{8}$$

where

$$\hat{A}_{I1}(\mathrm{t}) = \exp\left\{i\left(\omega_1 \hat{A}_1^+ \hat{A}_1 + \omega_2 \hat{A}_2^+ \hat{A}_2\right)t\right\} \times \hat{A}_1 \exp\left\{-i\left(\omega_1 \hat{A}_1^+ \hat{A}_1 + \omega_2 \hat{A}_2^+ \hat{A}_2\right)t\right\}$$
$$\hat{A}_{I2}(t) = \exp\left\{i(\omega_1 \hat{A}_1^+ \hat{A}_1 + \omega_2 \hat{A}_2^+ \hat{A}_2)t\right\} \times \hat{A}_2 \exp\left\{-i(\omega_1 \hat{A}_1^+ \hat{A}_1 + \omega_2 \hat{A}_2^+ \hat{A}_2)t\right\}$$
$$\hat{\sigma}_I^+(t) = e^{\frac{i\omega_0 \hat{\sigma}_z t}{2}} \hat{\sigma}^+ e^{\frac{-i\omega_0 \hat{\sigma}_z t}{2}} = \hat{\sigma}^+ e^{i\omega_0 t}$$
$$\hat{\sigma}_I^-(t) = e^{\frac{i\omega_0 \hat{\sigma}_z t}{2}} \hat{\sigma}^- e^{\frac{-i\omega_0 \hat{\sigma}_z t}{2}} = \hat{\sigma}^- e^{-i\omega_0 t}, \tag{9}$$

where we use $\left[\hat{\sigma}_\pm, \hat{\sigma}_z\right] = \mp 2\hat{\sigma}_\pm$ and the expansion [58]

$$\exp\left(\alpha\hat{A}\right)\hat{B}\exp\left(-\alpha\hat{A}\right) = \hat{B} + \alpha\left(\hat{A},\hat{B}\right) + \frac{\alpha^2}{2!}\left[\hat{A},\left(\hat{A},\hat{B}\right)\right] + \cdots, \tag{10}$$

The time evolution operator in the interaction picture reads (Dyson expansion)

$$\hat{U}_I(t,0) = \hat{T}\exp\left[-\frac{i}{\hbar}\int_0^t \hat{H}_I(t_1)dt_1\right], \tag{11}$$

where $\hat{T}$ is the time-ordering operator, which is a shorthand notation for the expansion

$$\begin{aligned}
&\hat{T}\exp\left[-\frac{i}{\hbar}\int_0^t \hat{H}_I(t_1)\,dt_1\right] = \\
&= 1 - \frac{i}{\hbar}\int_0^t \hat{H}_I(t_1)\,dt_1 + \\
&\quad + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}_I(t_1)\hat{H}_I(t_2) + \\
&\quad + \left(\frac{-i}{\hbar}\right)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3) + \\
&\quad + \cdots = \\
&= 1 - ig\int_0^t dt_1\left[\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1) + \hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\right] \times \\
&\quad \times \left[\hat{\sigma}^+ e^{i\omega_0 t_1} + \hat{\sigma}^- e^{-i\omega_0 t_1}\right] + \\
&+ (-ig)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \left[\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1) + \hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\right] \times \\
&\times \left[\hat{\sigma}^+ e^{i\omega_0 t_1} + \hat{\sigma}^- e^{-i\omega_0 t_1}\right] \\
&\times \left[\hat{A}_{I1}^+(t_2)\hat{A}_{I2}^+(t_2) + \hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\right] \times \\
&\left[\hat{\sigma}^+ e^{i\omega_0 t_2} + \hat{\sigma}^- e^{-i\omega_0 t_2}\right] + \cdots
\end{aligned} \tag{12}$$

The density operator of the total system $\hat{\rho}_I(t)$ is given by:

$$\hat{\rho}_I(t) = \hat{U}_I(t)\hat{\rho}_I(0)\hat{U}_I^+(t), \tag{13}$$

If we use relations (12) and (13), we have:

$$\hat{\rho}_I(t) = \left\{1 - \frac{i}{\hbar}\int_0^t \hat{H}_I(t_1)\,dt_1 + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}_I(t_1)\hat{H}_I(t_2) + \cdots\right\} \times$$

$$\times\hat{\rho}_I(0)\left\{1+\frac{i}{\hbar}\int_0^t \hat{H}_I(t_1)dt_1+ +\left(-\frac{i}{\hbar}\right)^2\int_0^t dt_1\int_0^{t_1}dt_2\hat{H}_I(t_1)\hat{H}_I(t_2)+\cdots\right\}=$$

$$=\left\{1-ig\int_0^t dt_1\begin{bmatrix}\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)\\+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\end{bmatrix}\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_1}+\hat{\sigma}^-e^{-i\omega_0 t_1}\right]+ +(-ig)^2\int_0^t dt_1\int_0^{t_1}dt_2\begin{bmatrix}\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)\\+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\end{bmatrix}\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_1}+\hat{\sigma}^-e^{-i\omega_0 t_1}\right]\times \times\left[\hat{A}_{I1}^+(t_2)\hat{A}_{I2}^+(t_2)+\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\right]\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_2}+\hat{\sigma}^-e^{-i\omega_0 t_2}\right]+\cdots\right\}\times$$

$$\times\hat{\rho}_I(0)\left\{1+ig\int_0^t dt_1\begin{bmatrix}\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)\\+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\end{bmatrix}\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_1}+\hat{\sigma}^-e^{-i\omega_0 t_1}\right]+ +(ig)^2\int_0^t dt_1\int_0^{t_1}dt_2\begin{bmatrix}\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)\\+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\end{bmatrix}\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_1}+\hat{\sigma}^-e^{-i\omega_0 t_1}\right]\times \times\left[\hat{A}_{I1}^+(t_2)\hat{A}_{I2}^+(t_2)+\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\right]\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_2}+\hat{\sigma}^-e^{-i\omega_0 t_2}\right]+\cdots\right\}=$$

$$=\hat{\rho}_I(0)+g^2\int_0^t dt_1\int_0^t dt_2\begin{bmatrix}\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)\hat{\rho}_I(0)\times\\\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)+\\+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_I(0)\times\\\times\hat{A}_{I1}^+(t_2)\hat{A}_{I2}^+(t_2)+\\+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\rho_I(0)\times\\\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\\+A_{I1}^+(t_1)A_{I2}^+(t_1)\hat{\rho}_I(0)+\\+A_{I1}^+(t_2)A_{I2}^+(t_2)\end{bmatrix}\times \times\left[\hat{\sigma}^+\hat{\sigma}^-e^{-i\omega_0(t_2-t_1)}+\hat{\sigma}^-\hat{\sigma}^+e^{i\omega_0(t_2-t_1)}\right]+ +ig\hat{\rho}_I(0)\int_0^t dt_2\begin{bmatrix}\hat{A}_{I1}^+(t_2)\hat{A}_{I2}^+(t_2)\\+\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\end{bmatrix}\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_2}+\hat{\sigma}^-e^{-i\omega_o t_2}\right]- -ig\int_0^t dt_1\left[\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\right]\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_1}+\hat{\sigma}^-e^{-i\omega_0 t_1}\right]\hat{\rho}_I(0)+O\left(g^3\right)+\cdots \tag{14}$$

where we have used $\left(\hat{\sigma}^+\right)^2=\left(\hat{\sigma}^-\right)^2=0$ and $\hat{H}_I(t)=\hat{H}_I^+(t)$. In section V, the coupling constant has been considered small $\left(\frac{g}{\omega_0}=0.01\right)$; therefore, we will ignore the sentences of g^3 and higher order in the following relations.

Now the reduced density matrix of the atom can be obtained by tracing out the cavity field degrees of freedom as follows:

$$\hat{\rho}_a(t)=Tr_f\left[\hat{\rho}_I(t)\right]=Tr_f\left[\hat{\rho}_I(0)\right]+ +g^2\int_0^t dt_1\int_0^t dt_2\begin{bmatrix}Tr_f[\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)\hat{\rho}_I(0)\\\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)]\\+Tr_f[\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_I(0)\\\times\hat{A}_{I1}^+(t_2)\hat{A}_{I2}^+(t_2)]\\+Tr_f[\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\rho_I(0)\\\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)]\\+Tr_f[A_{I1}^+(t_1)A_{I2}^+(t_1)\hat{\rho}_I(0)\\\times A_{I1}^+(t_2)A_{I2}^+(t_2)]\end{bmatrix}\times \times\left[\hat{\sigma}^+\hat{\sigma}^-e^{-i\omega_0(t_2-t_1)}+\hat{\sigma}^-\hat{\sigma}^+e^{i\omega_0(t_2-t_1)}\right]+ +ig\hat{\rho}_I(0)\int_0^t dt_2 Tr_f\begin{bmatrix}\hat{A}_{I1}^+(t_2)\hat{A}_{I2}^+(t_2)\\+\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\end{bmatrix}\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_2}+\hat{\sigma}^-e^{-i\omega_o t_2}\right]- -ig\int_0^t dt_1 Tr_f\left[\hat{A}_{I1}^+(t_1)\hat{A}_{I2}^+(t_1)+\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\right]\times \times\left[\hat{\sigma}^+e^{i\omega_0 t_1}+\hat{\sigma}^-e^{-i\omega_0 t_1}\right]\hat{\rho}_I(0) \tag{15}$$

Let the initial density matrix of the total system be a product state as follows:

$$\hat{\rho}_I(0)=\hat{\rho}_a(0)\otimes\hat{\rho}_f(0) \tag{16}$$

where $\hat{\rho}_a(0)$ is the initial density matrix of the two -level atom and $\hat{\rho}_f(0)$ is the initial density of cavity-field. We assume that the field is initially in a two-mode f-deformed coherent state $|z_1,z_2,f_1,f_2\rangle$,

$$\hat{\rho}_f(0)=|z_1,z_2,f_1,f_2\rangle\langle z_1,z_2,f_1,f_2|= =\sum_{n_1,n_2=0}^{\infty}C_{n_1}(z_1,f_1)C_{n_2}(z_2,f_2)\times \times C_{n_1}^*(z_1,f_1)C_{n_2}^*(z_2,f_2)|n_1,n_2\rangle\langle n_1,n_2| \tag{17}$$

where $C_{n_i}(z_i,f_i)=\frac{N_i z_i^{n_i}}{\sqrt{(n_i f_i^2(n_i))!}}$ (N_i is the normalization constant) and $z_i=|z_i|e^{i\varphi_i}$ [59].

The states $|z_1,z_2,f_1,f_2\rangle$ are defined as right eigenstates of the f-deformed annihilation operator $\hat{A}_1\hat{A}_2=\hat{a}_1 f_1(\hat{n}_1)\hat{a}_2 f_2(\hat{n}_2)$, *i. e.* $\hat{A}_1\hat{A}_2|z_1,z_2;f_1f_2\rangle=z_1z_2|z_1,z_2;f_1f_2\rangle$.

Now we use $Tr_f[\hat{\rho}_I(0)]=\hat{\rho}_a(0)$ and the following relations (see Appendix for details):

$$\begin{aligned}&Tr_f\left[\hat{A}^+_{I1}(t_1)\hat{A}^+_{I2}(t_1)\hat{\rho}_f(0)\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{A}^+_{I1}(t_1)\hat{A}^+_{I2}(t_1)\hat{\rho}_f(0)\times\\&\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)|n_1,n_2\rangle=\\&=\sum_{n_1,n_2=0}^{\infty}n_1n_2\left|C'_{n_1-1}(z_1,f_1)\right|^2\left|C'_{n_2-1}(z_2,f_2)\right|^2\times\\&\times\exp\{-i[\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)][t_2-t_1]\}\end{aligned}\tag{18}$$

$$\begin{aligned}&Tr_f\left[\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\hat{A}^+_{I1}(t_2)\hat{A}^+_{I2}(t_2)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\times\\&\times\hat{A}^+_{I1}(t_2)\hat{A}^+_{I2}(t_2)|n_1,n_2\rangle=\\&=\sum_{n_1,n_2=0}^{\infty}(n_1+1)(n_2+1)\times\\&\times\left|C'_{n_1+1}(z_1,f_1)\right|^2\left|C'_{n_2+1}(z_2,f_2)\right|^2\times\\&\times\exp\{i[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)][t_2-t_1]\}\end{aligned}\tag{19}$$

$$\begin{aligned}&Tr_f\left[\hat{A}^+_{I1}(t_1)\hat{A}^+_{I2}(t_1)\hat{\rho}_f(0)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{A}^+_{I1}(t_2)\hat{A}^+_{I2}(t_2)\hat{\rho}_f(0)|n_1,n_2\rangle=0\\&Tr_f\left[\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\hat{\rho}_f(0)|n_1,n_2\rangle=0\end{aligned}$$

$$\begin{aligned}&Tr_f\left[\hat{\rho}_f(0)\hat{A}^+_{I1}(t_1)\hat{A}^+_{I2}(t_1)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{\rho}_f(0)\hat{A}^+_{I1}(t_2)\hat{A}^+_{I2}(t_2)|n_1,n_2\rangle=0\\&Tr_f\left[\hat{\rho}_f(0)\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{\rho}_f(0)\hat{A}^+_{I1}(t_2)\hat{A}^+_{I2}(t_2)|n_1,n_2\rangle=0\end{aligned}\tag{20}$$

$$\begin{aligned}&Tr_f\left[\hat{A}^+_{I1}(t_1)\hat{A}^+_{I2}(t_1)\hat{\rho}_f(0)\hat{A}^+_{I1}(t_2)\hat{A}^+_{I2}(t_2)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{A}^+_{I1}(t_1)\hat{A}^+_{I2}(t_1)\hat{\rho}_f(0)\times\\&\times\hat{A}^+_{I1}(t_2)\hat{A}^+_{I2}(t_2)|n_1,n_2\rangle=0\\&Tr_f\left[\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\right]=\\&=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\times\\&\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)|n_1,n_2\rangle=0\end{aligned}\tag{21}$$

where we have defined

$$\begin{aligned}&\gamma_1(n_1)=(n_1+1)f_1^2(n_1+1)-n_1f_1^2(n_1),\\&\gamma_2(n_2)=(n_2+1)f_2^2(n_2+1)-n_2f_2^2(n_2),\\&\left|C'_{n_1-1}(z_1,f_1)\right|^2=f_1^2(n_1)\left|C_{n_1-1}(z_1,f_1)\right|^2,\\&\left|C'_{n_2-1}(z_2,f_2)\right|^2=f_2^2(n_2)\left|C_{n_2-1}(z_2,f_2)\right|^2,\\&\left|C'_{n_1}(z_1,f_1)\right|^2=f_1^2(n_1+1)\left|C_{n_1}(z_1,f_1)\right|^2,\\&\left|C'_{n_2}(z_2,f_2)\right|^2=f_2^2(n_2+1)\left|C_{n_2-1}(z_2,f_2)\right|^2\end{aligned}\tag{22}$$

Therefore, we obtain

$$\begin{aligned}&\hat{\rho}_a(t)=Tr_f\left[\hat{\rho}_I(t)\right]=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{\rho}_I(t)|n_1,n_2\rangle\\&=\hat{\rho}_a(0)+\\&+g^2\int_0^t dt_1\int_0^t dt_2\left[\begin{array}{l}\sum_{n_1,n_2=0}^{\infty}n_1n_2\left|C'_{n_1-1}(z_1,f_1)\right|^2\times\\\times\left|C'_{n_2-1}(z_2,f_2)\right|^2\\\times e^{-i[\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)][t_2-t_1]}\end{array}\right]\times\\&\times\hat{\rho}_a(0)\left[\hat{\sigma}^+\hat{\sigma}^-e^{-i\omega_0(t_2-t_1)}+\hat{\sigma}^-\hat{\sigma}^+e^{i\omega_0(t_2-t_1)}\right]+\\&+g^2\int_0^t dt_1\int_0^t dt_2\left[\begin{array}{l}\sum_{n_1,n_2=0}^{\infty}(n_1+1)(n_2+1)\times\\\times\left|C'_{n_1+1}(z_1,f_1)\right|^2\left|C'_{n_2+1}(z_2,f_2)\right|^2\times\\\times\exp\{i[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)][t_2-t_1]\}\end{array}\right]\times\\&\times\hat{\rho}_a(0)\left[\hat{\sigma}^+\hat{\sigma}^-e^{-i\omega_0(t_2-t_1)}+\hat{\sigma}^-\hat{\sigma}^+e^{i\omega_0(t_2-t_1)}\right]\end{aligned}\tag{23}$$

If we use the rotating wave approximation (with ignoring the term $\hat{A}^{+}_{I1}\hat{A}^{+}_{I2}\hat{\sigma}^{+}_{I} + \hat{A}_{I1}\hat{A}_{I2}\hat{\sigma}^{-}_{I}$ in the Eq. (8)), we have

$$\hat{H}'_I(t) = \hbar g\begin{bmatrix} \hat{A}^{+}_{I1}(t)\hat{A}^{+}_{I2}(t)\hat{\sigma}^{-}e^{-i\omega_0 t} \\ +\hat{A}_{I1}(t)\hat{A}_{I2}(t)\hat{\sigma}^{+}e^{i\omega_0 t} \end{bmatrix} \quad (24)$$

The time-evolution $\hat{U}'_I$ is given by:

$$\hat{U}'_I(t,0) = T\exp\left[-\frac{i}{\hbar}\int_0^t \hat{H}'_I(t)\,dt\right] =$$
$$= 1 - \frac{i}{\hbar}\int_0^t \hat{H}'_I(t_1)\,dt_1 +$$
$$+\left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}'_I(t_1)\hat{H}'_I(t_2) + \cdots =$$
$$= 1 - ig\int_0^t dt_1 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\sigma}^{-}e^{-i\omega_0 t_1} \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix} +$$
$$+(-ig)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1) \\ \times\hat{\sigma}^{-}e^{-i\omega_0 t_1} \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1) \\ \times\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix} \times$$
$$\times\begin{bmatrix} \hat{A}^{+}_{I1}(t_2)\hat{A}^{+}_{I2}(t_2)\hat{\sigma}^{-}e^{-i\omega_0 t_2} \\ +\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\hat{\sigma}^{+}e^{i\omega_0 t_2} \end{bmatrix} + \cdots \quad (25)$$

If we use the relation (13), we have

$$\rho'_I(t,0) = 1 - ig\int_0^t dt_1 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\sigma}^{-}e^{-i\omega_0 t_1} + \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix} +$$
$$+(-ig)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\sigma}^{-}e^{-i\omega_0 t_1} + \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix} \times$$
$$\times\begin{bmatrix} \hat{A}^{+}_{I1}(t_2)\hat{A}^{+}_{I2}(t_2)\hat{\sigma}^{-}e^{-i\omega_0 t_2} + \\ \hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\hat{\sigma}^{+}e^{i\omega_0 t_2} \end{bmatrix} + \cdots\} \times$$
$$\times\hat{\rho}_I(0)\left\{1 + ig\int_0^t dt_1 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\sigma}^{-}e^{-i\omega_0 t_1} + \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix} +\right.$$
$$+(ig)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\sigma}^{-}e^{-i\omega_0 t_1} + \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix} \times$$
$$\times\begin{bmatrix} \hat{A}^{+}_{I1}(t_2)\hat{A}^{+}_{I2}(t_2)\hat{\sigma}^{-}e^{-i\omega_0 t_2} \\ +\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\hat{\sigma}^{+}e^{i\omega_0 t_2} \end{bmatrix} + \cdots\} =$$
$$= \hat{\rho}_I(0) - -ig\int_0^t dt_1 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\sigma}^{-}e^{-i\omega_0 t_1} \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix}\hat{\rho}_I(0) +$$
$$+ig\hat{\rho}_I(0)\int_0^t dt_2 \begin{bmatrix} \hat{A}^{+}_{I1}(t_2)\hat{A}^{+}_{I2}(t_2)\hat{\sigma}^{-}e^{-i\omega_0 t_2} + \\ +\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\hat{\sigma}^{+}e^{i\omega_0 t_2} \end{bmatrix} +$$
$$+g^2\int_0^t dt_1 \int_0^t dt_2 \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\rho}_I(0)\hat{A}_{I1}(t_2)\times \\ \hat{A}_{I2}(t_2)\hat{\sigma}^{-}\hat{\sigma}^{+}e^{i\omega_0(t_2-t_1)} \end{bmatrix} +$$
$$+g^2\int_0^t dt_1 \int_0^t dt_2 \begin{bmatrix} \hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_I(0)\hat{A}^{+}_{I1}(t_2)\times \\ \times\hat{A}^{+}_{I2}(t_2)\hat{\sigma}^{+}\hat{\sigma}^{-}e^{-i\omega_0(t_2-t_1)} \end{bmatrix} +$$
$$+O(g^3) + \cdots \quad (26)$$

In this section, we will also ignore the sentences of g^3 and higher order in the following relations because the coupling constant has been considered small $\left(\frac{g}{\omega_0} = 0.01\right)$ as the state of without rotating wave approximation. By using Eq.(13, 15, 16, 18-22), we have

$$\hat{\rho}'_a(t) = Tr_f\left[\hat{\rho}'_I(t)\right] = \sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{\rho}'_I(t)|n_1,n_2\rangle =$$
$$= Tr_f\left(\hat{\rho}_I(0)\right) -$$
$$-ig\int_0^t dt_1 Tr_f \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\sigma}^{-}e^{-i\omega_0 t_1} \\ +\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\sigma}^{+}e^{i\omega_0 t_1} \end{bmatrix}\hat{\rho}_I(0) +$$
$$+ig\hat{\rho}_I(0)\int_0^t dt_2 Tr_f \begin{bmatrix} \hat{A}^{+}_{I1}(t_2)\hat{A}^{+}_{I2}(t_2)\hat{\sigma}^{-}e^{-i\omega_0 t_2} + \\ +\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\hat{\sigma}^{+}e^{i\omega_0 t_2} \end{bmatrix} +$$
$$+g^2\int_0^t dt_1 \int_0^t dt_2 Tr_f \begin{bmatrix} \hat{A}^{+}_{I1}(t_1)\hat{A}^{+}_{I2}(t_1)\hat{\rho}_I(0)\hat{A}_{I1}(t_2)\times \\ \times\hat{A}_{I2}(t_2)\hat{\sigma}^{-}\hat{\sigma}^{+}e^{i\omega_0(t_2-t_1)} \end{bmatrix} +$$
$$+g^2\int_0^t dt_1 \int_0^t dt_2 Tr_f \begin{bmatrix} \hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_I(0)\hat{A}^{+}_{I1}(t_2)\times \\ \times\hat{A}^{+}_{I2}(t_2)\hat{\sigma}^{+}\hat{\sigma}^{-}e^{-i\omega_0(t_2-t_1)} \end{bmatrix} =$$
$$= \hat{\rho}_a(0) +$$
$$+g^2\int_0^t dt_1 \int_0^t dt_2 \begin{bmatrix} \sum_{n_1,n_2=0}^{\infty} n_1 n_2 \\ \left|C'_{n_1-1}(z_1,f_1)\right|^2 \left|C'_{n_2-1}(z_2,f_2)\right|^2 \\ \times e^{-i[\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)][t_2-t_1]} \end{bmatrix} \times$$
$$\times\hat{\rho}_a(0)\left[\hat{\sigma}^{-}\hat{\sigma}^{+}e^{i\omega_0(t_2-t_1)}\right] +$$
$$+g^2\int_0^t dt_1 \int_0^t dt_2 \begin{bmatrix} \sum_{n_1,n_2=0}^{\infty} (n_1+1)(n_2+1) \\ \times\left|C'_{n_1+1}(z_1,f_1)\right|^2 \left|C'_{n_2+1}(z_2,f_2)\right|^2 \\ \times e^{i[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)][t_2-t_1]} \end{bmatrix} \times$$
$$\times\hat{\rho}_a(0)\left[\hat{\sigma}^{+}\hat{\sigma}^{-}e^{-i\omega_0(t_2-t_1)}\right] \quad (27)$$

We assume that the two-level atom is initially prepared in a coherent superposition of the excited state $|2\rangle$ and the ground state $|1\rangle$,

$$\hat{\rho}_a(0)=|\psi\rangle_{a\,a}\langle\psi|=\sum_{\ell,k=1,2}C'_\ell C'^*_k|\ell\rangle\langle k|,$$
$$|C'_1|^2+|C'_2|^2=1 \quad (28)$$

IV. Dynamical Properties of the Atom

In this section we shall study the effect of counter-rotating terms and parameter deformation on the various atomic properties including population inversion, the quantum fluctuation of atomic dipole variables and the atomic linear entropy.

A. *Atomic Population Inversion with and without the RWA*

Quantum interference in phase space is said to result in the revival of the atomic inversion [60]. Therefore, origin of this non-classical phenomenon is the quantum coherence resulted from the interaction between the cavity-field and the atom. Ionization detectors have been used to monitor the atomic beam exiting the cavity [61]. By using the reduced atomic density operator (15), in the absence of the RWA, the atomic population inversion at time t is obtained as follows:

$$W_1(t)=\langle\hat{\sigma}_z(t)\rangle=Tr_a\left[\hat{\rho}_a(t)\hat{\sigma}_z\right]=\langle 1|\hat{\rho}_a(t)\hat{\sigma}_z|1\rangle+$$
$$+\langle 2|\hat{\rho}_a(t)\hat{\sigma}_z|2\rangle=$$
$$=B+Bg^2\int_0^t dt_1\int_0^t dt_2\sum_{n_1,n_2=0}^{\infty}(n_1+1)(n_2+1)$$
$$\times\left|C_{n_1+1}(z_1,f_1)\right|^2\left|C_{n_{2+1}}(z_2,f_2)\right|^2$$
$$\times\exp\{i[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)-\omega_0][t_2-t_1]\}+$$
$$+Bg^2\int_0^t dt_1\int_0^t dt_2\sum_{n_1,n_2=0}^{\infty}n_1n_2\times$$
$$\times\left|C_{n_1-1}(z_1,f_1)\right|^2\left|C_{n_2-1}(z_2,f_2)\right|^2\times$$
$$\times\exp\{-i[\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)+\omega_0][t_2-t_1]\}$$
(29)

Here, we used $\hat{\sigma}_z=|2\rangle\langle 2|-|1\rangle\langle 1|$, $B=|C'_2|^2-|C'_1|^2$. By changing the integration variables as follows:

$$u=t_2-t_1$$
$$v=t_2+t_1$$
$$dudv=2dt_2dt_1 \quad (30)$$

We find:

$$W_1(t)=B\Bigg[1+2g^2t\sum_{n_1,n_2=0}^{\infty}(n_1+1)(n_2+1)\times$$
$$\times\left|C_{n_1+1}(z_1,f_1)\right|^2\left|C_{n_2+1}(z_2,f_2)\right|^2\times$$
$$\times\frac{Sin\left[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)-\omega_0\right]t}{\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)-\omega_0}+$$
$$+2g^2t\sum_{n_1,n_2=0}^{\infty}n_1n_2\left|C_{n_1-1}(z_1,f_1)\right|^2\left|C_{n_2-1}(z_2,f_2)\right|^2\times$$
$$\times\frac{Sin\left[\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)+\omega_0\right]t}{\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)+\omega_0}$$
(31)

If we use the relation (27), in the presence of the RWA, we have

$$W_2(t)=\langle\hat{\sigma}_z(t)\rangle=Tr_a\left[\hat{\rho}'_a(t)\hat{\sigma}_z\right]=\langle 1|\hat{\rho}'_a(t)\hat{\sigma}_z|1\rangle+$$
$$+\langle 2|\hat{\rho}'_a(t)\hat{\sigma}_z|2\rangle=B+$$
$$|C'_2|^2g^2\int_0^t dt_1\int_0^t dt_2\sum_{n_1,n_2=0}^{\infty}(n_1+1)(n_2+1)\times$$
$$\times\left|C_{n_{1+1}}(z_1,f_1)\right|^2\left|C_{n_2+1}(z_2,f_2)\right|^2\times$$
$$\times\exp\{i[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)-\omega_0][t_2-t_1]\}-$$
$$-|C'_1|^2g^2\int_0^t dt_1\int_0^t dt_2\sum_{n_1,n_2=0}^{\infty}n_1n_2\times$$
$$\times\left|C_{n_1-1}(z_1,f_1)\right|^2\left|C_{n_2-1}(z_2,f_2)\right|^2\times$$
$$\times\exp\{-i[\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)+\omega_0][t_2-t_1]\}$$
(32)

From (30) and it follows that:

$$W_2(t)=B++2g^2t\Bigg[\ |C'_2|^2\sum_{n_1,n_2=0}^{\infty}(n_1+1)(n_2+1)\times$$
$$\times\left|C_{n_2+1}(n_2+1)\right|^2\left|C_{n_{2+1}}(z_2,f_2)\right|^2\times$$
$$\times\frac{\sin\left[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)-\omega_0\right]t}{\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)-\omega_0}-$$
$$-|C'_1|^2\sum_{n_1,n_2=0}^{\infty}n_1n_2\left|C_{n_1-1}(z_1,f_1)\right|^2\left|C_{n_2-1}(z_2,f_2)\right|^2\times$$

$$\times\frac{\sin\left[\omega_1\gamma_1\left(n_1-1\right)+\omega_2\gamma_2\left(n_2-1\right)+\omega_0\right]t}{\omega_1\gamma_1\left(n_1-1\right)+\omega_2\gamma_2\left(n_2-1\right)+\omega_0}\Bigg] \quad (33)$$

In Fig. 1 we plot the population inversion $W_1(t)$ as a function of the scaled time $\omega_0 t$, in absence of the RWA, for an initially prepared excited atomic state $(C_1'=0, C_2'=1)$ interacting off-resonantly ($\frac{\omega_1}{\omega_0}=\frac{\omega_2}{\omega_0}=5$) with the two-mode cavity field in three cases:

(i) non-deformed case with $f_1(\hat{n}_1)=f_2(\hat{n}_2)=1$, q=1,

(ii) two-mode deformed case with

$$f_i(\hat{n}_i)=\sqrt{\frac{1}{\hat{n}_i}\frac{q^{\hat{n}_i}-1}{q-1}}, \quad q=0.9,\ i=1,2,$$

(iii) two-mode deformed case with

$$f_i(\hat{n}_i)=\sqrt{\frac{1}{\hat{n}_i}\frac{q^{\hat{n}_i}-1}{q-1}}, \quad q=1.1, i=1,2.$$

The deformation parameter q may be viewed as a phenomenological constant controlling the strength of the intensity-dependent coupling between the atom and the field. Furthermore, the choice of the nonlinearity function as $f(\hat{n})=\sqrt{\frac{1}{\hat{n}}\frac{q^{\hat{n}}-1}{q-1}}$ corresponds to the *maths-type* q-deformed coherent state [43]. In Fig. 2, we plot the population inversion $W_2(t)$ in the presence of the RWA with the same values as the rest of the parameter in Fig. 1. In the absence of the RWA, the atomic inversion for the degenerate two-photon non-deformed JCM oscillates around zero and amplitude of oscillations increases over time. We also see that atomic inversion for the non-degenerate two-photon deformed DJCM (q=1.1) shows slow oscillations in an irregular manner around zero. We also observe that in the deformed state (q=0.9), the collapse-revival phenomena can occur and the amplitude and the time of collapse and revival oscillations increases over time. In fact, the collapse-revival phenomena can occur with the decrease deformation parameter due to the deformed model under consideration. As Fig. 2 shows, the atomic inversion is always greater than the JCM and DJCM model without the rotating wave approximation $(W_2(t)>W_1(t))$. This means that in the JCM and DJCM models beyond the RWA, more energy is stored in the cavity field and the atom tends to remain in the ground state. This is caused by virtual-photon processes.

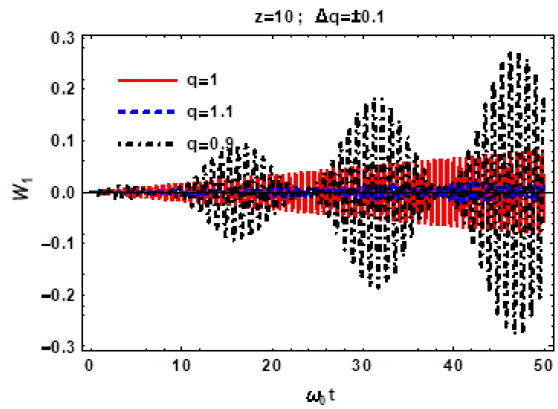

Fig. 1. Time evolution of the population inversion $W_1(t)$, in absence of the RWA, for a two-level atom initially prepared in the excited atomic state $(C_1'=0, C_2'=1)$ interacting off-resonantly with the state $\left|z_1,z_2;f_1f_2\right\rangle$ for two-mode cavity field in three cases (q=1, q=0.9, q=1.1). We have set $\left|z_1\right|=\left|z_2\right|=\left|z\right|=10, \frac{g}{\omega_0}=0.01\ \frac{\omega_1}{\omega_0}=\frac{\omega_2}{\omega_0}=5$

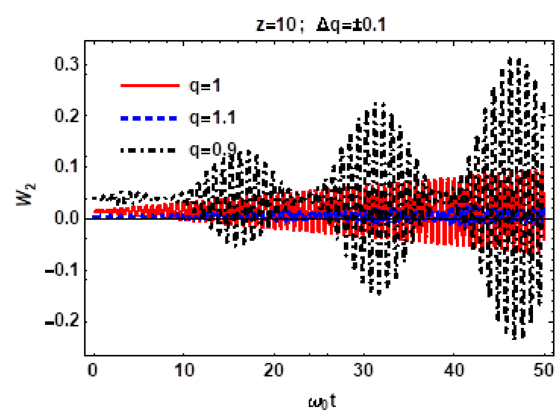

Fig. 2. Time evolution of the population inversion $W_2(t)$, in presence of the RWA, for a two-level atom initially prepared in the excited atomic state $(C_1' = 0,\ C_2' = 1)$ interacting off-resonantly with the state $|z_1, z_2; f_1 f_2\rangle$ for two-mode cavity field in three cases (q=1, q=0.9, q=1.1). We have set $|z_1| = |z_2| = |z| = 10, \frac{g}{\omega_0} = 0.01\ \frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_0} = 5$

B. *Atomic Dipole Squeezing*

The following slowly varying Hermitian quadrature operators were considered in this work in order to examine the quantum fluctuations of atomic dipole variables and to analyze the effect of CRTS on squeezing them

$$\hat{\sigma}_x(t) = \frac{1}{2}\left(\hat{\sigma}^+ e^{-i\omega_o t} + \hat{\sigma}^- e^{i\omega_o t}\right) \tag{34}$$

$$\hat{\sigma}_y(t) = \frac{1}{2i}\left(\hat{\sigma}^+ e^{-i\omega_o t} - \hat{\sigma}^- e^{i\omega_o t}\right) \tag{35}$$

The absorptive and dispersive components of the atomic polarization amplitude are represented by $\hat{\sigma}_y$ and $\hat{\sigma}_x$, respectively [62]. These components follow $[\hat{\sigma}_x, \hat{\sigma}_y] = i\hat{\sigma}_z/2$, which is the commutation relation. Thus, the uncertainty relation of Heisenberg is

$$\left\langle (\Delta\hat{\sigma}_x(t))^2 \right\rangle \left\langle (\Delta\hat{\sigma}_y(t))^2 \right\rangle \geq \frac{1}{16}\left|\langle \hat{\sigma}_z(t)\rangle\right|^2 \tag{36}$$

Here, $\left\langle (\Delta\hat{\sigma}_i(t))^2 \right\rangle = \left\langle \hat{\sigma}_i^2(t) \right\rangle - \left\langle \hat{\sigma}_i(t) \right\rangle^2$ is the variance in $\hat{\sigma}_i (i = x, y)$ of the atomic dipole.

If the variance in $\hat{\sigma}_i$ satisfies the condition, the component fluctuation for $\hat{\sigma}_i (i = x, y)$ is squeezed

$$\left\langle (\Delta\hat{\sigma}_i(t))^2 \right\rangle < \frac{1}{4}\left|\langle \hat{\sigma}_z(t)\rangle\right|, \quad (i=x \text{ or } y) \tag{37}$$

Because $\left\langle \hat{\sigma}_i^2(t) \right\rangle = \frac{1}{4}$ as the following will be resulted $F_i(t) = 1 - 4\langle \hat{\sigma}_i(t)\rangle^2 - \left|\langle \hat{\sigma}_z(t)\rangle\right| < 0$, ($i$=$x$ or y) (38)

Now, we assume that the two-level atom is initially in the state

$$|\psi\rangle_a = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle \tag{39}$$

If we use the relation (28), we obtain

$$C_1' = C_2' = \frac{1}{\sqrt{2}} \tag{40}$$

In the absence of the RWA, we can write

$$\left\langle \hat{\sigma}_{1x}(t) \right\rangle = Tr_a(\hat{\rho}_a(t)\hat{\sigma}_{1x}) = \langle 1|\hat{\rho}_a(t)\hat{\sigma}_{1x}|1\rangle + \langle 2|\hat{\rho}_a(t)\hat{\sigma}_{1x}|2\rangle \tag{41}$$

$$\left\langle \hat{\sigma}_{1y}(t) \right\rangle = Tr_a(\hat{\rho}_a(t)\hat{\sigma}_{1y}) = \langle 1|\hat{\rho}_a(t)\hat{\sigma}_{1y}|1\rangle + \langle 2|\hat{\rho}_a(t)\hat{\sigma}_{1y}|2\rangle \tag{42}$$

In the presence of the RWA, we have

$$\left\langle \hat{\sigma}_{1x}(t) \right\rangle = Tr_a(\hat{\rho}_a'(t)\hat{\sigma}_{1x}) = \langle 1|\hat{\rho}_a'(t)\hat{\sigma}_{1x}|1\rangle + + \langle 2|\hat{\rho}_a'(t)\hat{\sigma}_{1x}|2\rangle \tag{43}$$

$$\left\langle \hat{\sigma}_{1y}(t) \right\rangle = Tr_a(\hat{\rho}_a'(t)\hat{\sigma}_{1y}) = \langle 1|\hat{\rho}_a'(t)\hat{\sigma}_{1y}|1\rangle + + \langle 2|\hat{\rho}_a'(t)\hat{\sigma}_{1y}|2\rangle \tag{44}$$

In Fig. 3, we plot the time evolution of $F_{1x}(t)$ corresponding to the squeezing of $\sigma_{1x}(t)$, in the absence of the RWA, for an initially mixed atomic state ($C_1' = C_2' = \frac{1}{\sqrt{2}}$) interacting off-resonantly ($\frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_0} = 5$) with the cavity field under the three different aforesaid cases. As it can be seen, in the absence of the RWA, the function $F_{1x}(t)$ shows oscillation n an irregular manner around ($F_{1x}(t) = -1$) and $\sigma_{1x}(t)$ can be squeezed for the degenerate two-photon non-deformed JCM (q=1) and non-degenerate two-photon deformed DJCM (q=0.9) in most of the time regions. Furthermore, it is observed that the function $F_{1x}(t)$ shows oscillation in an irregular manner around ($F_{1x}(t) = -1$) and $\sigma_{1x}(t)$ can be squeezed for the non-degenerate two-photon deformed DJCM (q=1.1) almost at all the time.

We also observe that by decreasing the deformed parameter, the amplitude and the time of atomic dipole squeezing oscillations increases. Physically, it is due to the deformed model under consideration. In Fig. 4, we show the effect of the RWA on the temporal evolution of atomic dipole squeezing. As it can be seen, in the presence of the RWA, the function $F_{2x}(t)$ oscillation in a regular manner around ($F_{2x}(t)=-1$) and $\sigma_{2x}(t)$ can be squeezed for the degenerate two-photon non-deformed JCM (q=1) and the non-degenerate two-photon deformed DJCM (q=1.1, q=0.9) at all time. This means that the influence of counter-rotating terms leads to irregular oscillation atomic dipole squeezing and disappearance of dipole squeezing in some time regions. In fact, by comparing Fig.3 with Fig.4, we find that there exists destructive effect of virtual-photon on dipole squeezing. In Fig.5, we plot the time evolution of $F_{1y}(t)$ corresponding to the squeezing of $\sigma_{1y}(t)$, in the absence of the RWA, for an initially mixed atomic state ($C_1'=C_2'=\frac{1}{\sqrt{2}}$) interacting off-resonantly ($\frac{\omega_1}{\omega_0}=\frac{\omega_2}{\omega_0}=5$) with the cavity field under the three different aforesaid cases. As it can be seen, in the absence of the RWA, the function $F_{1y}(t)$ shows oscillation in an irregular manner around ($F_{1y}(t)=0$) and fluctuation in $\sigma_{1y}(t)$ is not squeezed for degenerate two-photon non-deformed JCM (q=1) and non-degenerate two-photon deformed DJCM (q=1.1, q=0.9) in most of the time regions. In Fig.6, we show the effect of the RWA on the temporal evolution of atomic dipole squeezing. As it can be seen, in the presence of the RWA, the function $F_{2y}(t)$ oscillation in a regular manner around ($F_{2y}(t)=0$) and $\sigma_{2y}(t)$ can be squeezed for the degenerate two-photon non-deformed JCM (q=1) and the non-degenerate two-photon deformed DJCM (q=1.1, q=0.9) in exactly half time range. This means that the influence of counter-rotating terms leads to the time of atomic dipole squeezing decreasing.

C. *Atomic Linear Entropy*

As a known fact, providing that the atom and the field in the JMC are originally made in a pure form, the atom-field system changes to an entangled form when t>0. In this state, the atom and the field individually are in mixed states. The stability of a pure state is resulted when the state's quantum coherence is maintained along its time evolution. Therefore, an initial state of pure quantum ($\hat{\rho}$) will be stable providing that $Tr\hat{\rho}^2=1$, at all times. The linear entropy [63] $S=1-Tr\hat{\rho}^2$ can be used to measure the stability. The time evolution of the atomic entropy replicates that of the entanglement degrees between the field and atom. Greater entropy will cause higher entanglement between the field and the atom.

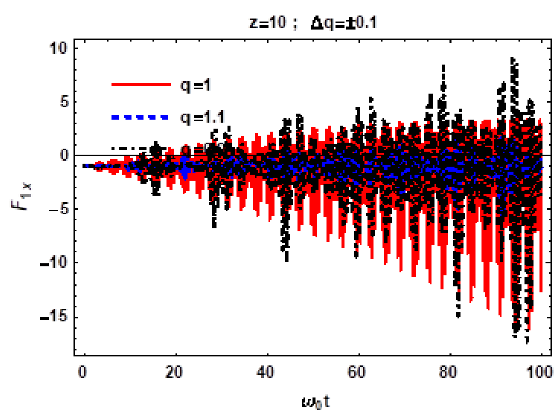

Fig. 3. Time evolution of $F_{1x}(t)$ corresponding to the squeezing of $\sigma_{1x}(t)$, in absence of the RWA, for a two-level atom initially prepared in a coherent superposition state ($C_1'=C_2'=\frac{1}{2}$) interacting off-resonantly with the state $|z_1,z_2;f_1f_2\rangle$ for two-mode cavity field in three cases (q=1, q=0.9,

q=1.1).We have set $|z_1| = |z_2| = |z| = 10$, $\frac{g}{\omega_0} = 0.01, \frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_0} = 5$

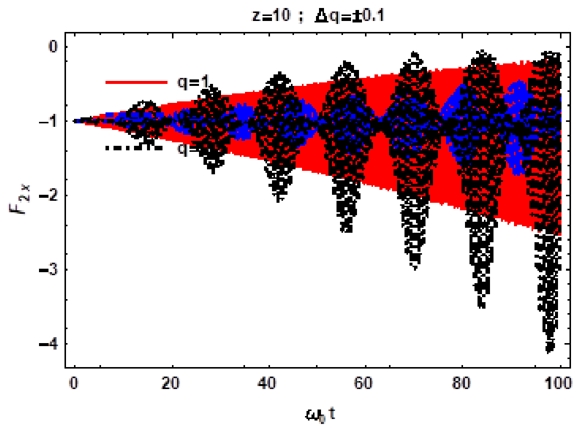

Fig. 4. Time evolution of $F_{2x}(t)$ corresponding to the squeezing of $\sigma_{2x}(t)$, in presence of the RWA, for a two-level atom initially prepared in a coherent superposition state $(C_1' = C_2' = \frac{1}{2})$ interacting off-resonantly with the state $|z_1, z_2; f_1 f_2\rangle$ for two-mode cavity field in three cases (q=1, q=0.9, q=1.1). We have set $|z_1| = |z_2| = |z| = 10, \frac{g}{\omega_0} = 0.01,$ $\frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_0} = 5$

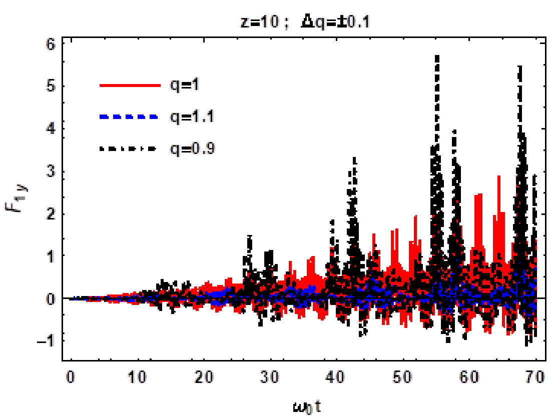

Fig. 5. Time evolution of $F_{1y}(t)$ corresponding to the squeezing of $\sigma_{1y}(t)$, in absence of the RWA, for a two-level atom initially prepared in a coherent superposition state $(C_1' = C_2' = \frac{1}{2})$ interacting off-resonantly with the state $|z_1, z_2; f_1 f_2\rangle$ for two-mode cavity field in three cases (q=1, q=0.9, q=1.1). We have set $|z_1| = |z_2| = |z| = 10$, $\frac{g}{\omega_0} = 0.01, \frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_0} = 5$

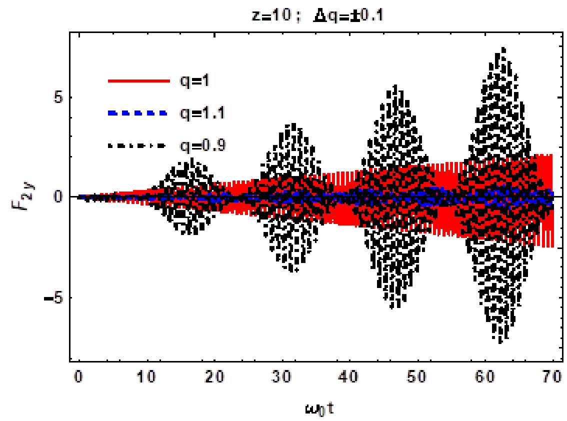

Fig. 6.Time evolution of $F_{2y}(t)$ corresponding to the squeezing of $\sigma_{2y}(t)$, in presence of the RWA, for a two-level atom initially prepared in a coherent superposition state $(C_1' = C_2' = \frac{1}{2})$ interacting off-resonantly with the state $|z_1, z_2; f_1 f_2\rangle$ for two-mode cavity field in three cases (q=1, q=0.9, q=1.1). We have set $|z_1| = |z_2| = |z| = 10, \frac{g}{\omega_0} = 0.01,$ $\frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_0} = 5$

In the absence of RWA, we can write

$$\hat{\rho}_a(t) = Tr_f\left[\hat{\rho}_I(t)\right] = \sum_{n_1, n_2 = 0}^{\infty} \langle n_1, n_2 | \hat{\rho}_I(t) | n_1, n_2 \rangle \tag{45}$$

Then, in the absence of RWA, the time evolution of the atomic linear entropy is given by

$$S_{1a} = 1 - Tr_a\left(\hat{\rho}_a^2(t)\right) \tag{46}$$

In the presence of RWA, we can write

$$\hat{\rho}_a'(t) = Tr_f\left[\hat{\rho}_I'(t)\right] = \sum_{n_1, n_2 = 0}^{\infty} \langle n_1, n_2 | \hat{\rho}_I'(t) | n_1, n_2 \rangle \tag{47}$$

Then, in the presence of RWA, the time evolution of the atomic linear entropy is given by

$$S_{2a} = 1 - Tr_a\left(\hat{\rho}_a'^2(t)\right) \tag{48}$$

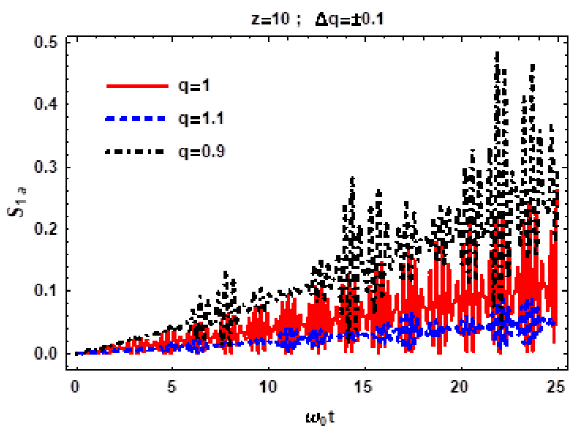

Fig. 7. Time evolution of the atomic linear entropy $S_{1a}(t)$, in absence of the RWA, for a two-level atom initially prepared in ground state $(C_1'=1, C_2'=0)$ interacting off-resonantly with the state $|z_1, z_2; f_1 f_2\rangle$ for two-mode cavity field in three cases (q=1, q=0.9, q=1.1). We have set $|z_1|=|z_2|=|z|=10, \frac{g}{\omega_0}=0.01, \frac{\omega_1}{\omega_0}=\frac{\omega_2}{\omega_0}=5$

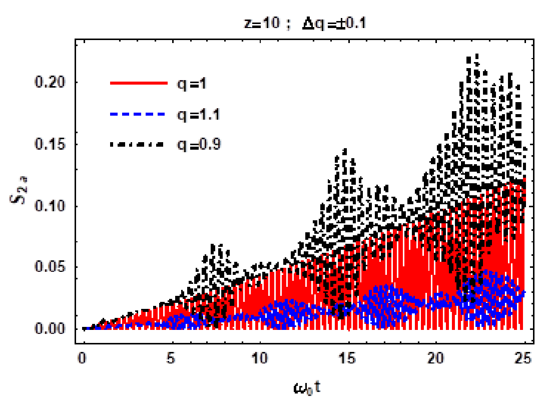

Fig. 8. Time evolution of the atomic linear entropy $S_{2a}(t)$, in presence of the RWA, for a two-level atom initially prepared in the ground state $(C_1'=1, C_2'=0)$ interacting off-resonantly with the state $|z_1, z_2; f_1 f_2\rangle$ for two-mode cavity field in three cases (q=1, q=0.9, q=1.1). We have set $|z_1|=|z_2|=|z|=10$, $\frac{g}{\omega_0}=0.01, \frac{\omega_1}{\omega_0}=\frac{\omega_2}{\omega_0}=5$

In Figs. 7 and 8, we plot the time evolution of the atomic linear entropy in the absence and presence of the RWA respectively (S_{1a}, S_{2a}). From Fig. 7, it follows that in the absence of the RWA, the atomic linear entropy in the two-photon non-deformed JCM behaves quasi-periodically with local maxima and minima in the course of time evolution. The presence of the local maxima and minima is, respectively, duo to the entanglement and disentanglement between the field and atom. For the non-degenerate two-photon DJCMs under consideration, the atomic linear entropy exhibits a chaotic behavior. We also observe that the maximum and the minimum atomic linear entropy amplitude were influenced by the deformation parameter. In fact, the minimum values of the entanglement between the atom and field can be increased by decreasing the deformation parameter. By comparing Fig. 7 with 8, we notice that entanglement amplitude decreases by the rotating-wave approximation. Physically, it is due to the influence of counter-rotating terms on the atomic linear entropy. This shows the significance of the effect of virtual-photon field on the atomic linear entropy. We observe in the two-photon non-deformed JCM (q=1) with and without the RWA, the maximum entanglement amplitude increases with the time which means that the atom and the cavity-field combined system can increase its initial maximal entangled state periodically.

V. Summary and Conclusion

In this paper, we studied dynamical properties of an effective two-level atom coupled to a two-mode f-deformed cavity field with and without the rotating wave approximation. Then, we investigated the effect of the counter-rotating term on temporal evolution of various non-classical properties of the atom, i.e., atomic population inversion, atomic dipole squeezing and atom-field entanglement. Particularly, we compared the numerical results for three different values of the deformation parameter q (q=1, q=1.1, q=0.9) with and without the rotating wave approximation. We observed in the absence of the RWA, the atomic inversion for the degenerate two-photon non-deformed JCM oscillates around zero and amplitude of oscillations increases over time. We also saw that atomic inversion for the non-degenerate two-photon deformed DJCM (q=1.1) shows slow oscillations in an irregular manner around zero. We also observed that in the deformed state (q=0.9), the collapse-revival phenomena can occur and the amplitude and the time of collapse and revival oscillations increases over time. Furthermore, it was observed that the atomic inversion is always greater than the JCM and DJCM model without the rotating wave approximation $(W_2(t) > W_1(t))$.

It was also seen that in the absence of the RWA, the function $F_{1x}(t)$ shows oscillation in an irregular manner around ($F_{1x}(t) = -1$) and $\sigma_{1x}(t)$ can be squeezed for the degenerate two-photon non-deformed JCM (q=1) and non-degenerate two-photon deformed DJCM (q=0.9) in most of the time regions. Furthermore it is observed that the function $F_{1x}(t)$ shows oscillation in an irregular manner around ($F_{1x}(t) = -1$) and $\sigma_{1x}(t)$ can be squeezed for the non-degenerate two-photon deformed DJCM (q=1.1) almost at all the time. It was also seen that in the presence of the RWA, the function $F_{2x}(t)$ oscillation in a regular manner around ($F_{2x}(t) = -1$) and $\sigma_{2x}(t)$ can be squeezed for the degenerate two-photon non-deformed JCM (q=1) and the non-degenerate two-photon deformed DJCM (q=1.1, q=0.9) at all time. It was also seen that in the absence of the RWA, the function $F_{1y}(t)$ shows oscillation in an irregular manner around ($F_{1y}(t) = 0$) and fluctuation in $\sigma_{1y}(t)$ is not squeezed for degenerate two-photon non-deformed JCM (q=1) and non-degenerate two-photon deformed DJCM (q=1.1, q=0.9) in most of the time regions. It was also seen that in the presence of the RWA, the function $F_{2y}(t)$ oscillation in a regular manner around ($F_{2y}(t) = 0$) and $\sigma_{2y}(t)$ can be squeezed for the degenerate two-photon non-deformed JCM (q=1) and the degenerate two-photon deformed DJCM (q=1.1, q=0.9) in exactly half time range. It was also observed that in the absence of the RWA, the atomic linear entropy in the two-photon non-deformed JCM behaves quasi-periodically with local maxima and minima in the course of time evolution. The presence of the local maxima and minima is, respectively, due to the entanglement and disentanglement between the field and atom. For the non-degenerate two-photon DJCMs under consideration, the atomic linear entropy exhibited a chaotic behavior. As it was seen, the maximum and the minimum atomic linear entropy amplitude were influenced by the deformation parameter. In addition, we observed that both factors, quantum deformation and counter-rotating terms, reinforce each other in the occurrence of nonlinear effects.

APPENDIX

In this appendix we prove the Eq.(18-21). First we prove the Eq.(18)

$$Tr_f\left[\hat{A}_{I1}^{+}(t_1)\hat{A}_{I2}^{+}(t_1)\hat{\rho}_f(0)\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\right]=$$

$$=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|\hat{A}_{I1}^{+}(t_1)\hat{A}_{I2}^{+}(t_1)\hat{\rho}_f(0)\times$$

$$\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)|n_1,n_2\rangle=$$

$$=\sum_{n_1,n_2=0}^{\infty}\langle n_1,n_2|e^{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_1}\hat{A}_1^{+}\times$$

$$\times\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$

$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$

$$\times\hat{A}_2^{+}\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_{21}\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$

$$\times\sum_{n_1',n_2'=0}^{\infty} C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)C_{n_1'}^{*}\left(z_1,f_1\right)\times$$
$$\times C_{n_2'}^{*}\left(z_2,f_2\right)\left|n_1',n_2'\right\rangle\left\langle n_1',n_2'\right|$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_2\right\}\times$$
$$\times\hat{A}_1\ \exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\hat{A}_2\ \exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\left|n_1,n_2\right\rangle$$
$$=\sum_{n_1,n_2=0}^{\infty}\ \sum_{n_1',n_2'=0}^{\infty}\sqrt{n_1n_2\left(n_1'+1\right)\left(n_2'+1\right)}\times$$
$$\times f_1\left(n_1\right)f_2\left(n_2\right)f_1\left(n_1'+1\right)f_2\left(n_2'+1\right)\times$$
$$\times C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\times$$
$$\times\exp\left\{i\omega_1[\left(n_1'+1\right)f_1^2\left(n_1'+1\right)-n_1'f_1^2\left(n_1'\right)]t_1\right\}\times$$
$$\times\exp\left\{i\omega_2\left[\left(n_2'+1\right)f_1^2\left(n_2'+1\right)-n_2'f_2^2\left(n_2'\right)\right]t_1\right\}\times$$
$$\times\delta_{n_1,n_1'+1}\delta_{n_2,n_2'+1}\times$$
$$\times\exp\left\{-i\omega_1[n_1f_1^2\left(n_1\right)-\left(n_1-1\right)f_1^2\left(n_1-1\right)]t_2\right\}\times$$
$$\times\exp\left\{-i\omega_2\left[n_2f_1^2\left(n_2\right)-\left(n_2-1\right)f_2^2\left(n_2-1\right)\right]t_2\right\}\times$$
$$\times\delta_{n_1',n_1-1}\delta_{n_2',n_2-1}$$
$$=\sum_{n_1,n_2=0}^{\infty}n_1n_2\left|C_{n_1-1}'(z_1,f_1)\right|^2\left|C_{n_2-1}'(z_2,f_2)\right|^2\times$$
$$\times\exp\left\{-i[\omega_1\gamma_1(n_1-1)+\omega_2\gamma_2(n_2-1)][t_2-t_1]\right\} \tag{A.1}$$

where we have used the Eq.(4, 9, 22) . Now we prove the relation (19)

$$Tr_f\left[\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\hat{A}_{I1}^{+}(t_2)\hat{A}_{I2}^{+}(t_2)\right]=$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\right|\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\times$$
$$\times\hat{A}_{I1}^{+}(t_2)\hat{A}_{I2}^{+}(t_2)\left|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\right|\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_1\right\}\hat{A}_1\times$$
$$\times\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\hat{A}_2\ \exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\sum_{n_1',n_2'=0}^{\infty}C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\left|n_1',n_2'\right\rangle\left\langle n_1',n_2'\right|\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_2\right\}\times$$
$$\times\hat{A}_1^{+}\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\hat{A}_2^{+}\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\left|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\ \sum_{n_1',n_2'=0}^{\infty}\sqrt{\left(n_1+1\right)\left(n_2+1\right)n_1'n_2'}\times$$
$$\times f_1\left(n_1\right)f_2\left(n_2\right)f_1\left(n_1'\right)f_2'\left(n_2'\right)C_{n_1'}\left(z_1,f_1\right)\times$$
$$\times C_{n_2'}\left(z_2,f_2\right)C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\times$$
$$\times\exp\left\{i\omega_1[\left(n_1'-1\right)f_1^2\left(n_1'-1\right)-n_1'f_1^2\left(n_1'\right)]t_1\right\}\times$$
$$\times\exp\left\{i\omega_2\left[\left(n_2'-1\right)f_1^2\left(n_2'-1\right)-n_2'f_2^2\left(n_2'\right)\right]t_1\right\}\times$$
$$\times\delta_{n_1,n_1'-1}\delta_{n_2,n_2'-1}\times$$
$$\times\exp\left\{i\omega_1[\left(n_1+1\right)f_1^2\left(n_1+1\right)-n_1f_1^2\left(n_1\right)]t_2\right\}\times$$
$$\times\exp\left\{i\omega_2\left[\left(n_2+1\right)f_1^2\left(n_2+1\right)-n_2f_2^2\left(n_2\right)\right]t_2\right\}\times$$
$$\times\delta_{n_1',n_1+1}\delta_{n_2',n_2+1}=\sum_{n_1,n_2=0}^{\infty}(n_1+1)(n_2+1)\times$$
$$\times\left|C_{n_1+1}'(z_1,f_1)\right|^2\left|C_{n_2+1}'(z_2,f_2)\right|^2\times$$
$$\times\exp\left\{i[\omega_1\gamma_1(n_1)+\omega_2\gamma_2(n_2)][t_2-t_1]\right\} \tag{A.2}$$

where we have used the Eq.(4, 9, 22). Now we prove the Eq. (20)

$$Tr_f\left[\hat{A}_{I1}^{+}(t_1)\hat{A}_{I2}^{+}(t_1)\hat{\rho}_f(0)\right]=$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\right|\hat{A}_{I1}^{+}(t_2)\hat{A}_{I2}^{+}(t_2)\hat{\rho}_f(0)\left|n_1,n_2\right\rangle$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\right|\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_1\right\}\hat{A}_1^{+}$$
$$\times\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\hat{A}_2^{+}\times$$
$$\times\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\sum_{n_1',n_2'=0}^{\infty}C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\left|n_1',n_2'\right\rangle\left\langle n_1',n_2'\right|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\sum_{n_1',n_2'=0}^{\infty}\sqrt{\left(n_1'+1\right)\left(n_2'+1\right)}\times$$
$$\times f_1\left(n_1'+1\right)f_2\left(n_2'+1\right)\times$$
$$\times C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\times$$
$$\times\exp\left\{i\omega_1\left[\left(n_1'+1\right)f_1^2\left(n_1'+1\right)-n_1'f_1^2\left(n_1'\right)\right]t_1\right\}\times$$
$$\times\exp\left\{i\omega_2\left[\left(n_2'+1\right)f_1^2\left(n_2'+1\right)-n_2'f_2^2\left(n_2'\right)\right]t_1\right\}\times$$
$$\times\delta_{n_1,n_1'+1}\delta_{n_2,n_2'+1}\delta_{n_1',n_1}\delta_{n_2',n_2}=0 \tag{A.3}$$

and

$$Tr_f\left[\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\right]=$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\left|\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\hat{\rho}_f(0)\right|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\right|\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_1\right\}\times$$
$$\times\hat{A}_1\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\hat{A}_2\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\sum_{n_1',n_2'=0}^{\infty}C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\times$$
$$\times\left|n_1',n_2'\right\rangle\left\langle n_1',n_2'\right|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\sum_{n_1',n_2'=0}^{\infty}\sqrt{n_1'n_2'}f_1\left(n_1'-1\right)f_2\left(n_2'-1\right)\times$$
$$\times C_{n_1'}\left(z_1,f_1\right)\times C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\times$$
$$\times\exp\left\{i\omega_1[\left(n_1'-1\right)f_1^2\left(n_1'-1\right)-n_1'f_1^2\left(n_1'\right)]t_1\right\}\times$$
$$\times\exp\left\{i\omega_2\left[\left(n_2'-1\right)f_1^2\left(n_2'-1\right)-n_2'f_2^2\left(n_2'\right)\right]t_1\right\}\times$$
$$\times\delta_{n_1,n_1'-1}\delta_{n_2,n_2'-1}\delta_{n_1',n_1}\delta_{n_2',n_2}=0 \qquad \text{(A.4)}$$

Now we prove the Eq. (21)

$$\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\left|\hat{A}_{I1}^{+}(t_1)\hat{A}_{I2}^{+}(t_1)\hat{\rho}_f(0)\times\right.\right.$$
$$\left.\times\hat{A}_{I1}^{+}(t_2)\hat{A}_{I2}^{+}(t_2)\right|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\right|e^{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_1}\hat{A}_1^{+}\times$$
$$\times\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\hat{A}_2^{+}\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\sum_{n_1',n_2'=0}^{\infty}C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\left|n_1',n_2'\right\rangle\left\langle n_1',n_2'\right|\times$$
$$\times\hat{A}_1^{+}\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_2\right\}\times$$
$$\times\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\hat{A}_2^{+}\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\left|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\sum_{n_1',n_2'=0}^{\infty}\sqrt{\left(n_1+1\right)\left(n_2+1\right)\left(n_1'+1\right)\left(n_2'+1\right)}\times$$
$$\times f_1\left(n_1+1\right)\times f_2\left(n_2+1\right)f_1\left(n_1'+1\right)\times$$
$$\times f_2'\left(n_2'+1\right)C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\times$$
$$\times\exp\left\{i\omega_1\left[\left(n_1'+1\right)f_1^2\left(n_1'+1\right)-n_1'f_1^2\left(n_1'\right)\right]t_1\right\}\times$$
$$\times\exp\left\{i\omega_2\left[\left(n_2'+1\right)f_1^2\left(n_2'+1\right)-n_2'f_2^2\left(n_2'\right)\right]t_1\right\}\times$$
$$\times\delta_{n_1,n_1'+1}\delta_{n_2,n_2'+1}\times$$
$$\times\exp\left\{i\omega_1\left[\left(n_1+1\right)f_1^2\left(n_1+1\right)-n_1f_1^2\left(n_1\right)\right]t_2\right\}\times$$
$$\times\exp\left\{i\omega_2\left[\left(n_2+1\right)f_1^2\left(n_2+1\right)-n_2f_2^2\left(n_2\right)\right]t_2\right\}\times$$
$$\times\delta_{n_1',n_1+1}\delta_{n_2',n_2+1}=0 \qquad \text{(A.5)}$$

and

$$\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\left|\hat{A}_{I1}(t_1)\hat{A}_{I2}(t_1)\hat{\rho}_f(0)\times\right.\right.$$
$$\left.\times\hat{A}_{I1}(t_2)\hat{A}_{I2}(t_2)\right|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\left\langle n_1,n_2\right|\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_1\right\}\times$$
$$\times\hat{A}_1\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\hat{A}_2\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_1\right\}\times$$
$$\times\sum_{n_1',n_2'=0}^{\infty}C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\left|n_1',n_2'\right\rangle\left\langle n_1',n_2'\right|\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}A_2\right)t_2\right\}\times$$
$$\times\hat{A}_1\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\exp\left\{i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\times$$
$$\times\hat{A}_2\exp\left\{-i\left(\omega_1\hat{A}_1^{+}\hat{A}_1+\omega_2\hat{A}_2^{+}\hat{A}_2\right)t_2\right\}\left|n_1,n_2\right\rangle=$$
$$=\sum_{n_1,n_2=0}^{\infty}\sum_{n_1',n_2'=0}^{\infty}\sqrt{n_1n_2n_1'n_2'}f_1\left(n_1\right)f_2\left(n_2\right)\times$$
$$\times f_1\left(n_1'\right)f_2'\left(n_2'\right)\times$$
$$\times C_{n_1'}\left(z_1,f_1\right)C_{n_2'}\left(z_2,f_2\right)\times$$
$$\times C_{n_1'}^{*}\left(z_1,f_1\right)C_{n_2'}^{*}\left(z_2,f_2\right)\times$$
$$\times\exp\left\{i\omega_1\left[\left(n_1'-1\right)f_1^2\left(n_1'-1\right)-n_1'f_1^2\left(n_1'\right)\right]t_1\right\}\times$$
$$\exp\left\{i\omega_2\left[\left(n_2'-1\right)f_1^2\left(n_2'-1\right)-n_2'f_2^2\left(n_2'\right)\right]t_1\right\}\times$$
$$\times\delta_{n_1,n_1'-1}\delta_{n_2,n_2'-1}$$

$$\times \exp\left\{i\omega_1\left[\left(n_1-1\right)f_1^2\left(n_1-1\right)-n_1f_1^2\left(n_1\right)\right]t_2\right\}\times$$
$$\times \exp\left\{i\omega_2\left[\left(n_2-1\right)f_1^2\left(n_2-1\right)-n_2f_2^2\left(n_2\right)\right]t_2\right\}\times$$
$$\times\delta_{n_1',n_1-1}\delta_{n_2',n_2-1}=0 \qquad (A.6)$$

REFERENCES

[1] H. Walter, B.T.H. Varcoe, B.G. Englert, and T. Becker, "Cavity quantum electro dynamics," Reports Progress Phys. Vol. 69, pp. 1325-1382, 2006.

[2] A. Blais, R.S. Huang, A. Wallraff, S.M. Girvin, and R.J. Schoelkopf, "Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation," Phys. Rev. A, Vol. 69, pp. 062320 (1-14), 2004.

[3] I.I. Rabi, "Space Quantization in a Gyrating Magnetic Field," Phys. Rev, Vol. 49, pp. 324-328, (1936); Vol. 51, pp. 652-654, 1937.

[4] E.T. Jaynes and F.W. Cummings, "Comparison of quantum and semiclassical radiation theories with application to the beam maser," Proc. IEEE, Vol. 51, pp. 89-109, 1963.

[5] L. Allen and J.H. Eberly, *Optical Resonance and Two-Level Atoms*, Dover Publications, 1987.

[6] G.S. Agarwal and R.R. Puri, "Exact quantum-electrodynamics results for scattering, emission, and absorption from a Rydberg atom in a cavity with arbitrary Q," Phys. Rev. A, Vol. 33, pp. 1757-1764, 1986.

[7] S.N. Barnett and P.L. Knight, "Dissipation in a fundamental model of quantum optical resonance," Phys. Rev. A, Vol. 33, pp. 2444-2448, 1986.

[8] J. Eislet and H. Risken, "Quasiprobability distributions for the cavity-damped Jaynes-Cummings model with an additional Kerr medium," Phys. Rev. A. Vol. 43, pp. 346-356, 1991.

[9] M.J. Werner and H. Risken, "Quasiprobability distributions for the cavity-damped Jaynes-Cummings model with an additional Kerr medium," Phys. Rev. Vol. 44, pp. 4623-4632, 1991.

[10] J.G. Peixoto de Faria and M.C. Nemes, "Dissipative dynamics of the Jaynes-Cummings model in the dispersive approximation: Analytical results," Phys. Rev. A, Vol. 59, pp. 3918-3925, 1999.

[11] M. Boissonneault, J.M. Gambetta, and A. Blais, "Nonlinear dispersive regime of cavity QED: The dressed dephasing model," Phys. Rev. A. Vol. **77**, pp. 060305(1-4), 2008.

[12] H.J. Carmichael and D.F. Walls, "Modifications to the Scully-Lamb laser master equation," J. Phys. A, Vol. 6, pp. 1552-1564, 1973.

[13] J.D. Cresser, "Thermal equilibrium in the Jaynes-Cummings model," J. Mod. Opt. Vol. 39, pp. 2187-2192, 1992.

[14] H. Zoubi, M. Orenstien, and A. Ron, "Coupled micro cavities with dissipation," Phys. Rev. A. Vol. 62, pp. 033801 (1-10), 2000.

[15] M. Scala, B. Militello, A. Messina, J. Piilo, and S. Maniscalco, "Microscopic derivation of the Jaynes-Cummings model with cavity losses," Phys. Rev. A, Vol. 75, pp. 013811 (1-8), 2007.

[16] F. Beaudoin, J.M. Gambetta, and A. Blais, "Dissipation and ultrastrong coupling in circuit QED," Phys. Rev. A, Vol. 84, pp. 043832 (1-15), 2011.

[17] C.W. Duncan and A. del. Campo, "Shortcuts to Adiabaticity Assisted by Counter adiabatic Born-Oppenheimer Dynamics," New J. Phys. Vol. 20, pp. 085003 (1-14), 2018.

[18] M.H. Naderi, "The Jaynes–Cummings model beyond the rotating-wave approximation as an intensity-dependent model: quantum statistical and phase properties," J. Phys. A: Math. Theor. Vol. 44, pp. 055304 (1-20). 2011.

[19] T. Niemczyk, F. Deppe, H. Huebl, E.P. Menzel, H. Hocke, M.J. Schwarz, J.J. García-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx, and R. Gross, "Circuit Quantum Electrodynamics in the Ultra-strong-coupling Regime," Nature Phys. Vol. 6, pp. 772-776, 2010.

[20] P. Forn-Diaz, J. Lisenfeld, D. Marcos, J.J. Garcia-Ripoll, E. Solano, C.J.P.M. Harmans, and J.E. Mooij, "Observation of the Bloch-Siegert Shift in a Qubit-Oscillator System in the Ultrastrong Coupling Regime," Phys. Rev. Lett. Vol. 105, pp. 237001 (1-4), 2010.

[21] A. Fedorov, A.K. Feofanov, P. Macha, P. Forn-Diaz, C.J.P.M. Harmans, and J.E. Mooij, "Strong Coupling of a Quantum Oscillator to a Flux Qubit at Its Symmetry Point," Phys. Rev. Lett. Vol. 105, pp. 060503 (1-4), 2010.

[22] J. Hausinger and M. Grifoni,"Dissipative dynamics of a biased qubit coupled to a harmonic oscillator: analytical results beyond therotating wave approximation," New J. Phys. Vol. 10, pp. 115015 (1-39), 2008.

[23] S. De Liberato, D. Gerace, I. Carusotto, and C. Ciuti, "Extracavity quantum vacuum radiation from a single qubit," Phys. Rev. Lett. Vol. 80, pp. 053810 (1-5), 2009.

[24] X. Cao, J.Q. You, H. Zheng, and F. Nori, "A qubit strongly coupled to a resonant cavity: asymmetry of the spontaneous emission spectrum beyond the rotating wave approximation," New J. Phys. Vol. 13, pp. 073002 (1-12), 2011.

[25] A. Ridolfo, M. Leib, S. Savasta, and M.J. Hartmann, "Photon Blockade in the Ultrastrong Coupling Regime," Phys. Rev. Lett. Vol, 109, pp. 193602 (1-7), 2012.

[26] B. Buck and C.V. Sukumar,"Multi-phonon generalisation of the Jaynes-Cummings model," Phys. Lett. A, Vol. 81, pp. 132-135, 1981.

[27] V. Buzek, "Jaynes-Cummings model with intensity-dependent coupling interacting with Holstein-Primakoff SU(1,1) coherent state," Phys.Rev. A, Vol. 39, pp. 3196-3199, 1989.

[28] M. Chaichian, D. Ellinas, and P. Kulish, "Quantum algebra as the dynamical symmetry of the deformed Jaynes-Cummings model," Phys. Rev. Lett. Vol. 65, pp. 980-983, 1990.

[29] B.W. Shore and P. Knight, "The Jaynes-Cummings Model," J. Mod. Opt. Vol. 40, pp. 1195-1238, 1993.

[30] Z. Tang, "Approach to a generalized Jaynes model and the geometric phase," Phys. Rev. A, Vol 52, pp. 3448-3451, 1995.

[31] A.A. Budini, R.L. de Matos Filho, and N. Zagury, "Dissipation and decoherence in the generalized Jaynes-Cummings model," Phys. Rev. A, Vol. 67, pp. 0338 15 (1-21), 2003.

[32] A. Messina, S. Maniscalco, and A. Napoli, "Interaction of bimodal fields with few-level atoms in cavities and traps," J. Mod. Opt. Vol. 50, pp. 1-49, 2003.

[33] M. Abdel-Aty "Mixed-state entanglement of a three-level system with an intensity-dependent interaction," J. Phys. A: Math. Gen, Vol. 37, pp. 1759-1773, 2004.

[34] M.H. Naderi, M. Soltanolkotabi, and R. Roknizadeh, "Dynamical Properties of a Two-Level Atom in Three Variants of the Two-Photon *q*-Deformed Jaynes–Cummings Model," J. Phys. Soc. Jpn. Vol. 73, pp. 2413-2423, 2004.

[35] M.H. Naderi, M. Soltanolkotabi, and R. Roknizadeh, "A theoretical scheme for generation of nonlinear coherent states in a micromaser under intensity-dependent Jaynes-Cummings model," Eur. Phys. J. D, Vol. 32, pp. 397–408, 2005.

[36] R.L. de Matos Filho, and W. Vogel, "Nonlinear coherent states," Phys. Rev. A, Vol. 54, PP. 4560-4563, 1996.

[37] H. Grinberg, "Beyond the rotating wave approximation. An intensity dependent nonlinear coupling model in two-level systems," Phys. Lett. A, Vol. 374, pp. 1481-1487, 2010.

[38] M. Jimbo, "A q difference analog of U(g) and the Yang-Baxter equation," Lett. Math. Phys, Vol. 10, pp. 63-69,1985.

[39] A.J. Macfarlane, "On q-analogues of the quantum harmonic oscillator and the quantum group SU(2)," J. Phys. A: Math. Gen. Vol. 22, pp. 4581-4589, 1989.

[40] J. Crnugelj, M. Martinis, and V. Mikuta-Martinis, "Dynamical model of deformed coherent states for multiplicity distributions and correlations," Phys. Lett. B, Vol. 318, pp. 227-230, 1993.

[41] V. Buzek, "Jaynes-Cummings model with intensity-dependent coupling interacting with Holstein-Primakoff SU(1,1) coherent state," Phys. Rev. A. Vol. 39, pp. 3196-3199, 1989.

[42] M. Chaichian, D. Ellinas, and P. Kulish, "Quantum algebra as the dynamical symmetry of the deformed Jaynes-Cummings model," Phys. Rev. Lett. Vol. 65, pp. 980-983, 1990.

[43] M. Arik and D.D. Coon, "Hilbert spaces of analytic functions and generalized coherent states," J. Math. Phys. Vol. 17, pp. 524-527, 1976.

[44] D. Bonatsos, C. Daskaloyannis, and G.A. Lalazissis, "Unification of Jaynes-Cummings models," Phys. Rev. A, Vol. 47, pp. 3448-3451, 1993.

[45] A. Dehghani, B. Mojaveri, S. Shirin, and S. Amiri Faseghandis, "Parity Deformed Jaynes-Cummings Model Robust Maximally Entangled States," Sci. Rep. Vol. 10, pp. 1038-1053, 2016.

[46] R.L. de Matos Filho and W. Vogel, "Nonlinear coherent states," Phys. Rev. A, Vol. 54, pp. 4560-4563,1996.

[47] V.I. Monko, G. Marmo, E.C.G. Sudarshan, and F. Zaccaria, "Oscillators and Nonlinear Coherence States," Phys. Scr. Vol. 55, pp. 528-566, 1997.

[48] P.P. Munhoz and A. Vidiealla-Barronco, "Non-conditioned generation of Schroedinger cat states in a cavity," J. Mod. Opt. Vol. 52, pp. 1557-1573, 2005.

[49] J. Batouli, M. El Baz, and A. Maaouni, "RLC circuit realization of a q- deformed harmonic oscillator with time dependent mass," Phys. Lett. A, Vol. 13, PP. 1619-1626, 2015.

[50] M.H. Naderi, M. Soltanolkotabi, "Influence of nonlinear quantum dissipation on the dynamical properties of the f-deformed Jaynes-Cummings model in the dispersive limit," Eur. Phys. J. D, Vol. 39, pp. 471-479, 2006.

[51] M. Daeimohammad, F. Kheirandish, and M.R. Abolhasany, "Quantum Dynamics of a Dissipative Deformed Harmonic Oscillator," Int. J. Theor. Phys. Vol. 48, pp. 693-699, 2009.

[52] M. Daeimohammad, "Quantum dynamics of a two-mode *f*-deformed cavity field in a heat bath," Phys. Scr, Vol. 86, pp.065006 (1-7), 2012.

[53] M. Daeimohammad, F. Kheirandish, and K. Saeedi, "Quantum Dynamics of a Harmonic Oscillator in a Deformed Bath," Int. J. Theor. Phys. Vol. 50, pp. 171-180, 2011.

[54] M. Daeimohammad, "Influence of the counter-rotating terms on the quantum dynamics of the damped harmonic oscillator in a deformed bath," Int. J. Mod. Phys. B, Vol. 33, pp. 1950126 (1-13), 2019.

[55] S.C. Gou, "Quantum behavior of a two-level atom interacting with two modes of light in a cavity," Phys. Rev. A, Vol. 40, pp. 5116-5128, 1989.

[56] B. Buck and C.V. Sukumar, "Exact results for the emission from one and two atoms in an ideal cavity at multiphoton resonance," J. Phys. A: Math. Gen. Vol. 17, pp. 885-894, 1984.

[57] S.S. Sharma, N.K. Sharma, and L. Zamick, "Single two-level ion in an anharmonic-oscillator trap: Time evolution of the Q function and population inversion," Phys. Rev. A, Vol. 56, pp. 694-699, 1997.

[58] J. Wei and E. Norman, "Lie Algebraic Solution of Linear Differential Equations," J. Math. Phys. (N.Y.), Vol. 4, pp. 575-581, 1 963.

[59] M.H. Naderi, "Intrinsic decoherence effects on quantum dynamics of the nondegenerate two-photon f-deformed Jaynes-Cummings model governed by the Milburn equation," Can. J. Phys. Vol. 85, pp. 1071-1096, 2007.

[60] H.I. Yoo, J.J. Sanchez-Mondragon, and J.H. Eberly, "Non-linear dynamics of the fermion-boson model: interference between revivals and the transition to irregularity," J. Phys. A, Vol. 14, pp. 1383-1397, 1981.

[61] D. Meschede, H. Walther, and G. Muller, "One-Atom Maser," Phys. Rev. Lett. Vol. 54, pp. 551-554, 1985.

[62] S.M. Barnett, "General criterion for squeezing," Opt. Commun. Vol. 61, pp.432-466, 1987.

[63] W.H. Zurek, S. Habib, and J.P. Paz, "Coherent states via decoherence," Phys. Rev. Lett. Vol. 70, pp. 1187-1190, 1993.

Mohsen Daeimohammad received B.Sc. degree in physics from University of Isfahan, Iran, and M.Sc and PhD in atomic and molecular physics from Islamic Azad University, Science and Research Branch, Tehran, Iran. He is currently a senior assistant professor at Islamic Azad University, Najafabad Branch, Iran. His main research interests are Quantum Optics, Quantum Deformation, and Quantum theory of damping. He can be reached at m.daeimohammad@pco.iaun.ac.ir.

THIS PAGE IS INTENTIONALLY LEFT BLANK.