Coherent Transport of Single Photon in a Quantum Super-cavity with Mirrors Composed of A-Type Three-level Atomic Ensembles

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Abstract— In this paper, we study the coherent transport of single photon in a coupled resonator waveguide (CRW) where two three-level Λ-type atomic ensembles are embedded in two separate cavities. We show that it is possible to control the photon transmission and reflection coefficients by using classical control fields. In particular, we find that the total photon transmission and reflection are achievable. In addition, the two atomic ensembles can act as controllable mirrors of a secondary cavity (super-cavity) which represents localized photon states and makes it possible to store and retrieve single photon in the region sandwiched between the two atomic ensembles.

KEYWORDS: Coupled resonator waveguide (CRW), quantum super-cavity, single-photon transport, three-level atomic ensemble.

I. INTRODUCTION

Since the innovation of quantum information protocols such as quantum cryptography [1], quantum teleportation [2], and quantum communication [3] several systems have been proposed to implement quantum networks [4]. Among all the proposals, quantum optical systems have attracted considerable attention. On one hand, photons are ideal carriers of quantum information because they are fast, readily available, and robust against decoherence. On the other hand, atoms represent reliable and long-lived storage and processing units [5].

Up to now, many single-photon sources, such as semiconductor nano-crystals [6] and quantum dots [7], have been suggested and experimentally realized. However, in order to use single photon to transfer quantum information between different nodes of a quantum network it is necessary to find techniques and instruments for coherently control of single-photon transport and storage. In recent years many efforts have been exerted to implement devices that can control photons by photon-photon interaction [8, 9] or via nonlinear media [10]. Furthermore, the control of photon transport by an additional classical field in an artificial medium has been studied [11]. On the other hand, many methods have been proposed to realize quantum memories. For instance, photon storage and photon retrieval based on photon echo [12] and electromagnetically induced transparency (EIT) [13] have been investigated.

One of the systems that has recently been considered for storage and coherent transport of single photons is one dimensional coupled-resonator waveguide (CRW) [14] doped with two- or three-level atomic systems [15-17]. Coupled resonator waveguide is a type of waveguide based on coupling of individual resonators through evanescent fields [14] and can be realized by using the coupled defect modes in photonic crystals [18] or coupled superconducting line resonators [19]. A theoretical study on scattering of a single photon from a two-level atom in a one dimensional (1D) CRW [15] shows that single-photon transport can be controlled by manipulating the atomic transition frequency. Particularly, at the resonance condition the atom acts as a perfect mirror and reflects the photon completely. Inspired by this result, Zhou et al. proposed [17] a quantum analogue for the Fabry-Perot cavity, the so-called...
quantum super-cavity which is composed of two two-level atoms embedded in two separate cavities of a CRW. At the resonance condition the two atoms act as two highly reflecting mirrors and form a secondary cavity (super-cavity) which can confine the photon. Thus, not only the coherent control of single-photon transport is possible but also the photon can be stored and retrieved in the region sandwiched between the two atoms by controlling the transition frequencies of the two atoms.

With the purpose of generalizing this idea, in this paper we study the discrete scattering [15] of a single photon in a 1D CRW where two non-interacting $\Lambda$-type three-level atomic ensembles are located in two separate cavities of CRW. We show that the transmission spectrum of a single photon can be controlled by adjusting the oscillation frequency and the Rabi frequency of a classical external field. Besides, the investigation of quasi-bound states reveals that in the absence of dissipation processes and under some certain conditions a perfect super-cavity appears. Therefore, one can store and retrieve a photon in this cavity by adjusting the classical control field.

This paper is organized as follow: in section II we introduce the physical model of the system under consideration. In section III we investigate the transmission spectrum of a single photon. In section IV we study the quasi-bound states of the system and derive the conditions under which a super-cavity is formed. Finally, we summarize our conclusions in section V.

**II. PHYSICAL MODEL**

According to Fig. 1, we consider a 1D CRW which consists of $N$ identical single-mode cavities with resonance frequency $\omega_0$. We assume $N$ is large enough that periodic boundary condition becomes reasonable. Each cavity is weakly coupled to only nearest neighbors by hopping constant $\xi$.

![Fig. 1 Schematic diagram of a CRW doped by two $\Lambda$-type three-level atomic ensembles in $xd$ cavities. Each ensemble is coupled to the quantum field of the corresponding cavity and a classical control field.](image)

Therefore, we can use the tight-binding Hamiltonian [15], which is the photonic analogue of the tight-binding approximation in solid state physics [20], to describe the CRW ($\hbar = 1$):

$$\hat{H}_c = \omega_0 \sum_j \hat{a}_j^\dagger \hat{a}_j - \xi \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \text{h.c.})$$

where $\hat{a}_j^\dagger (\hat{a}_j)$ is the creation (annihilation) operator of the $j$th cavity mode. Because of the translational symmetry of CRW the Hamiltonian (1) can be diagonalized by using the Fourier transform,

$$\hat{a}_k = \frac{1}{\sqrt{NL}} \sum_j e^{ikjL} \hat{a}_j,$$

where $L$ is the lattice constant. So we have

$$\hat{H}_c = \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k,$$

with nonlinear dispersion relation

$$\omega_k = \omega_0 - 2\xi \cos kl,$$

which is similar to the dispersion relation of an ideal electronic crystal [20] and defines an energy band, $[\omega_0 - 2\xi, \omega_0 + 2\xi]$, for the photon propagation in the CRW.

Furthermore, the two non-interacting $\Lambda$-type three-level atomic ensembles are located in two separate cavities of the CRW. For simplicity, we choose the central cavity of the CRW as the coordinate-axis origin. The first ensemble with $N_1$ identical atoms and the second with $N_2$ identical atoms are located at

$$\Omega_a, \Omega_c$$

where $\Omega_a$ and $\Omega_c$ are the control fields.
the \(d\)-th cavity and \(d\)-th cavity, respectively. In order to implement such a system, natural and artificial \(\Lambda\)-type atoms [21] or three-level Josephson junctions [22] can be used. We denote the ground state, the meta-stable state and the excited state of the \(i\)-th atom in the \(l\)-th atomic ensemble as \(|\gamma_i^{(l)}\rangle\), \(|\alpha_i^{(l)}\rangle\) and \(|\epsilon_i^{(l)}\rangle\) \((l = 1, 2)\), respectively. As is shown in Fig. 1, the \(|\epsilon_i^{(l)}\rangle \rightarrow |\gamma_i^{(l)}\rangle\) transition is coupled to its corresponding cavity field and a strong classical field with the oscillation frequency \(\nu_c^{(l)}\) and the Rabi frequency \(\Omega_i\) matches to the \(|\epsilon_i^{(l)}\rangle \rightarrow |\alpha_i^{(l)}\rangle\) transition of all atoms in the \(l\)-th ensemble. Thus, the free Hamiltonian and the interaction Hamiltonian of atoms are, respectively, as follows:

\[
\hat{H}_0 = \sum_{i=1,2} \sum_{l=1}^{N_l} (\omega_e^{(l)} |\epsilon_i^{(l)}\rangle \langle \epsilon| + \omega_a^{(l)} |\alpha_i^{(l)}\rangle \langle \alpha|),
\]

\[
\hat{H}_I = \sum_{i=1,2} \sum_{l=1}^{N_l} (g_i^{(l)} (\hat{a}_{\epsilon^{-d,i}}^{+})|\gamma_i^{(l)}\rangle \langle \epsilon| + h.c.) + \Omega_l (e^{-i\nu_c^{(l)} t} |\alpha_i^{(l)}\rangle \langle \epsilon| + h.c.),
\]

where \(\omega_e^{(l)}\) and \(\omega_a^{(l)}\) represent the excited and the meta-stable frequencies of the \(l\)-th atomic ensemble. The interaction Hamiltonian is written in the rotating wave approximation and \(g_i^{(l)}\) is the coupling constant of the \(i\)-th atom of the \(l\)-th ensemble to its corresponding cavity field, that we assume is equal for all atoms of each ensemble: \(g_i^{(l)} = g_i\). Consequently, the total Hamiltonian of the system reads:

\[
\hat{H} = \hat{H}_0 + \hat{H}_a + \hat{H}_I.
\]

In order to omit the explicit time dependence of the total Hamiltonian we transform it into a rotating frame of reference which is defined by the unitary transformation \(\hat{U} = \exp(-i\hat{H}_d t)\)

where

\[
\hat{N} = \sum_j \hat{a}_j^{+} \hat{a}_j + \sum_{i=1,2} \sum_{l=1}^{N_l} (|\epsilon_i^{(l)}\rangle \langle \epsilon| + |\alpha_i^{(l)}\rangle \langle \alpha|).
\]
commutes with the total Hamiltonian \( \left[ \hat{H}, \hat{N} \right] = 0 \) i.e., the total excitation of the system is a constant of motion. Therefore, in order to study the coherent scattering of single photon it is sufficient to consider only one excitation subspace.

III. TRANSMISSION SPECTRUM

A. Scattering equation

To investigate the transmission of a single photon through the system, we consider the most general stationary eigenstate of the system in one excitation subspace:

\[
|E_k\rangle = \sum_j u_k(j) \hat{a}_j^\dagger |0\rangle + u_k^{(i)} |0\rangle + u_k^{(2)} |0\rangle,
\]

where \( u_k(j) \) is the probability amplitude for finding the photon at the \( j \)th cavity of the CRW. The first number 0 inside the kets shows the vacuum state of the CRW field. The state \( |e\rangle, (|a\rangle) \) represents a Dicke state \([23]\) of the \( l \)th ensemble that one of its atoms is in the excited (meta-stable) state and \( u_k^{(i)}(u_k^{(i)}) \) is the corresponding probability amplitude.

By using Eq. (12) in the eigenvalue equation \( \hat{H} |E_k\rangle = E_k |E_k\rangle \), we arrive at the following discrete scattering equation for \( u_k(j) \):

\[
[E_k - \omega_0 - \sum_{j=1,2} V_l(j)] u_k(j) = -\xi [u_k(j + 1) + u_k(j - 1)],
\]

where

\[
V_l(j) = \frac{g_i^2 N_j (E_k - \Delta^{(i)})}{(E_k - \omega^{(i)})(E_k - \Delta^{(i)}) - \Omega_j} \delta_{j,d-1/d}.
\]

Furthermore, we obtain:

\[
u^{(i)} = \frac{g_i \sqrt{N_j} (E_k - \Delta^{(i)})}{(E_k - \omega^{(i)})(E_k - \Delta^{(i)}) - \Omega_j} \xi u_k((-1)^d),
\]

\[
u^{(i)} = \frac{g_i \sqrt{N_j} \Omega_j}{(E_k - \omega^{(i)})(E_k - \Delta^{(i)}) - \Omega_j} \xi u_k((-1)^d).\]

We can consider Eq. (13) as a Schrödinger equation in which \( u_k(j) \) can be interpreted as a wave function for photon. In this manner, Eq. (13) shows that each atomic ensemble acts as a delta potential, \( V_l \), which is energy-dependent and its strength is related to the parameters of the classical field, i.e., \( \nu^{(i)} \) and \( \Omega_j \), so it can be controlled by adjusting these parameters. Depending on the relation among the photon energy, the parameters of the classical field and the atomic transition energies the potential may be attractive or repulsive.

B. Transmission coefficient

In order to determine the single-photon transmission coefficient we consider an elastic scattering of a single photon which comes from the left with the energy \( \omega_k \) and wave number \( k \). Since the potential \( V_l(j) \) is zero except at the points \( j = -d \) and \( d \) the wave function of the photon can be written in terms of the asymptotic plane waves as follows

\[
u_k(j) = \begin{cases} t \nu_{k}^{(l)} + r \nu_{k}^{(l)} & j < -d, \\ t \nu_{k}^{(l)} - r \nu_{k}^{(l)} & -d < j < d, \\ t \nu_{k}^{(l)} & j > d, \end{cases}
\]

where \( r \) and \( t \) denote the reflection and transmission amplitudes of photon, respectively and \( t_1 \) and \( r_1 \) show the probability amplitudes for finding the right-going and the left-going photons in the region sandwiched between the two atomic ensembles. Considering the continuity condition, \( u_k(\pm d^-) = u_k(\pm d^+), \) and the scattering equation (Eq. 13) at \( j = -d \) and \( j = d \) results in the following transmission amplitude for photon:

\[
u_k^{(l)} = \frac{g_i \sqrt{N_j} (E_k - \Delta^{(i)})}{(E_k - \omega^{(i)})(E_k - \Delta^{(i)}) - \Omega_j} \xi u_k((-1)^d),
\]

\[
u_k^{(i)} = \frac{g_i \sqrt{N_j} \Omega_j}{(E_k - \omega^{(i)})(E_k - \Delta^{(i)}) - \Omega_j} \xi u_k((-1)^d).\]
\[ t = 4\xi^2 \sin^2 kL \left( e^{i4kLd} - 1 \right) \left[ 4\xi^2 \sin^2 kLe^{i4kLd} + \left[ f_i(k) - V'e^{i4kLd} \right] \left[ f_i(k) - V'e^{i4kLd} \right] \right]^{-1}, \]

with \( f_i = V_1 - 2i\xi \sin kL \).

It is clear that by controlling the potential via changing \( \nu_1^{(i)} \) or \( \Omega_i \), one can control the transmission coefficient. In Fig. 2 we plot the transmission coefficient versus the wave number of the incident photon. In all these plots when \( E_k = \Delta^{(1)} = \Delta^{(2)} \) the transmission coefficient is equal to 1. This total transmission is due to the electromagnetically induced transparency (EIT) phenomenon [24].

Indeed, the quantum interference between the two atomic transition channels causes transparency of the medium for the incoming photon and the elimination of the back travelling light. In addition, since the hopping constant has the role of guidance for photon in the CRW and \( g_i \) shows the strength of the atomic potential, an increase in the ratio \( g_i / 2\xi \) leads to a decrease in the transmission coefficient. This fact can be seen by comparison of Fig. 2a and Fig. 2b.

Another important point is that for some values of the wave number the transmission coefficient is zero. The zero transmission at \( k = 0, \pm \pi \) is due to the energy band of the CRW and it is independent of \( g_i \). As we pointed before (Eq. 4), the periodicity of the CRW results in an allowed energy band for the propagation of light which acts as a photon filter such that a single photon with an energy outside this band do not interact with CRW and do not propagate through it. The mentioned wave numbers represent the boundaries of the energy band (see Eq. 4). Thus the group velocity of photons with those wave numbers is zero.

However, Fig. 2c shows that EIT can overcome to the energy band and photons that in the absence of ensembles could not propagate in CRW, now totally transmit...
through EIT phenomenon. Other total reflections occur at singularities of the potential. For these wave numbers the ensemble forms an infinite potential which prevents the transmission of photon. It should be pointed out that the oscillations shown in the transmission spectrum arise from the multiple interference in the region sandwiched between the two atomic ensembles and they become less visible by decreasing the distance $2d$.

**IV. QUANTUM SUPER-CAVITY**

In this section we investigate the possibility of formation the quantum super-cavity by considering the quasi-bound states of the system.

**A. Quasi-Bound State**

As we found in the previous section each atomic ensemble may act as a potential barrier, so there is a potential double barrier which can produce localized states in the space [25]. Photons can leak out of the sandwiched region between the two atomic ensembles owing to the finite width and height of the potential barriers. Therefore, the corresponding localized state is called a quasi-bound state or a resonant state. A resonant state is an eigenfunction of the Hamiltonian under the boundary condition that we have only outgoing waves [26]. Therefore, we consider the following solution for Eq.(13)

$$u_k(j) = \begin{cases} 
A e^{-ikLj} & j < -d \\
C e^{ikLj} & j > d \\
e^{ikLj} + B e^{-ikLj} & -d < j < d 
\end{cases}$$  \hspace{1cm} (7)$$

The scattering equation (Eq. 13) together with the continuity condition at $j = -d$ and $j = d$ impose the following condition for the existence of the state given by Eq. (18)

$$e^{i2kdL} = \pm \frac{f_1(k)f_2(k)}{V_1V_2}.$$  \hspace{1cm} (8)$$

Thus if we prepare a photon in the sandwiched region between the two atomic ensembles with a wave number $k$ that satisfies Eq. (19), its stationary state is a quasi-bound state as Eq. (18).

**B. Quantum super-cavity**

Now, for simplicity, we assume that the two atomic ensembles are identical: $V_1 = V_2 = V$ and $f_1(k) = f_2(k) = f(k)$. Thus Eq. (19) takes the form:

$$e^{2ikdL} = \pm \frac{f(k)}{V}.$$  \hspace{1cm} (9)$$

In this case, the system is symmetric with respect to the origin so the wave functions have definite parity. The plus and minus signs in Eq. (20) correspond to the wave functions with even and odd parity, respectively. By considering this condition, $u_k(j)$ (Eq. 18) can be written as follows for even and odd parity, respectively

- For odd parity:
  $$u_{k}^{\text{odd}}(j) = \begin{cases} 
-sin(kLd)e^{i kl(j-d)} & j < -d \\
sin(kLj) & -d < j < d \\
sin(kLd)e^{i kl(j+d)} & j > d 
\end{cases}$$  \hspace{1cm} (10)$$

- For even parity:
  $$u_{k}^{\text{even}}(j) = \begin{cases} 
cos(kLd)e^{i kl(j-d)} & j < -d \\
\cos(kLj) & -d < j < d \\
cos(kLd)e^{i kl(j+d)} & j > d 
\end{cases}$$  \hspace{1cm} (11)$$

For real wave numbers, Eq. (20) leads to the following equations

$$\cos(2k_{re}Ld) = \mp 1,$$
$$\sin(2k_{re}Ld) = \pm \frac{2\xi \sin(k_{re}L)}{V}.$$  \hspace{1cm} (12)$$

that are satisfied simultaneously when

$$k_{re} = q_n \quad (n = \text{integer})$$

$$q_n = \begin{cases} 
2n\pi / 2dL & \text{for odd parity} \\
(2n+1)\pi / 2dL & \text{for even parity} 
\end{cases}$$  \hspace{1cm} (13)$$

and

$$(\omega - 2\xi \cos q_n - \omega_c)(\omega - 2\xi \cos q_n - \Delta) = \Omega^2.$$  \hspace{1cm} (14)$$
Under these circumstances, the probability for finding the photon outside the sandwiched region is zero (Eq. 21 and Eq. 22). Indeed Eq. (25) represents the singularities of the potential and when it is satisfied there are two infinite barriers. If the distance between these barriers is an integer (odd parity) or half-integer (even parity) multiple of the wavelength of photon, a standing wave is formed and the photon is trapped between the barriers. In this case, the ensembles act as controllable mirrors of a cavity (super-cavity) where the photon can be stored inside it by adjusting the oscillation frequency or the Rabi frequency of the classical field. Figure 3 represents the probability $|\mu_k(j)|^2$ versus the cavity number and the Rabi frequency for an odd parity wave function. It can be seen that when the condition given by Eq. (25) is satisfied the photon is completely confined in the super-cavity.

**V. CONCLUSIONS**

By a theoretical study of the discrete 1D scattering of a single photon from two $\Lambda$-type atomic ensembles we showed that by adjusting the oscillation frequency and the Rabi frequency of the classical field one can effectively control the transmission spectrum of a single photon. In particular, the total reflection and transmission is achievable and the system can act as a quantum switch. We also showed that the two atomic ensembles can form a cavity with a controllable photon leakage which provides a method for storing and retrieving single photon.

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