Super operator Technique in Investigation of the Dynamics of a Two Non-Interacting Qubit System Coupled to a Thermal Reservoir

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ABSTRACT— In this paper, we clarify the applicability of the super operator technique for describing the dissipative quantum dynamics of a system consists of two qubits coupled with a thermal bath at finite temperature. By using super operator technique, we solve the master equation and find the matrix elements of the density operator.
Considering the qubits to be initially prepared in a general mixed state, we explore the influence of dissipation on dynamical behavior of system. As expected, we obtain analytical evidence that the thermal photons and spontaneous emission of atoms are responsible for dissipation of entanglement of system.

KEYWORDS: Decoherence, Super operator technique, thermal reservoir

I. INTRODUCTION

It is now well understood that entanglement is the key resource for implementation of many quantum information protocols like teleportation, cryptography and quantum communications [1]–[3].

Bi-partite entanglement between two quantum bits (quantum mechanical two level systems analogous to classical bits), has been found to be particularly important in this context. These quantum systems in interaction with an environment display an irreversible and dissipative dynamics.

To investigate the dynamics of the quantum dissipative systems for atom-field interactions, master equations are usually studied by using Fokker–Planck and Langevin equations. These approaches to the problem make it usually difficult to apply the solutions to an arbitrary initial field in contrast with the super operator techniques [4]. Moreover, for atomic systems master equations are usually solved by using Laplace transformation in which we need to find the solution to the set of coupled differential equations for matrix elements. It is generally a hard task to list up all of the equations and if one selects the basis carelessly, many of the equations look coupled to each other, even though they are not essentially so on the appropriate basis.

For dissipative finite-level systems, typical master equations can be solved in a compact form without resort to writing it down as a set of coupled equations among matrix elements and the solution is then naturally in an operator form.

Super operators differ from conventional operators in that they only act on each other and the density matrix. Their action on the wave function is undefined. It should be stressed that in this approach solution of master equation is based on the algebra obeyed by the superoperators that such equation contains. This will be achieved through a disentangling of an exponential of the sum of super operators which is deeply linked to the solvability of the algebra.

In this paper, the applicability of the super operator method is explained for a system consists of two non-interacting qubits coupled to a thermal reservoir. We present a procedure for solving master equations directly in an operator form.

The approach is initially sketched in Section II by defining three super operators [5] and rewriting master equation based on them. Then In Section III we solve the master equation in a compact form and obtain matrix elements of the density operator. In Section IV, we derive concurrence under the influence of thermal environment. Finally, we summarize our conclusions in Section V.

II. GENERAL PROCEDURE

In this section, we consider a system consists of 2 non-interacting qubits in a thermal bath. The time evolution of the density operator gives us information about the dynamics of the dissipative system. The master equation for 2 non-interacting qubits in the interaction picture has the following form

$$d\hat{\rho}/dt=\sum_{i=1}^{2}\Big[\gamma n\left(2\hat{\sigma}_i^{+}\hat{\rho}\hat{\sigma}_i^{-}-\hat{\sigma}_i^{-}\hat{\sigma}_i^{+}\hat{\rho}-\hat{\rho}\hat{\sigma}_i^{-}\hat{\sigma}_i^{+}\right)+\gamma(n+1)\left(2\hat{\sigma}_i^{-}\hat{\rho}\hat{\sigma}_i^{+}-\hat{\sigma}_i^{+}\hat{\sigma}_i^{-}\hat{\rho}-\hat{\rho}\hat{\sigma}_i^{+}\hat{\sigma}_i^{-}\right)\Big], \tag{1}$$

where $\hat{\rho}$ is the reduced density operator of atomic system. We suppose that two qubits are spatially well separated so we can neglect some collective effects such as dipole–dipole interaction and coherent interaction between two qubits in a common bath. For reservoir, n denotes the mean photon numbers of the thermal bath coupling to qubits and γ represents spontaneous emission of atoms.

In Eq. (1) $\hat{\sigma}_i^{+}$ and $\hat{\sigma}_i^{-}$ with $i=1,2$ correspond raising and lowering operators of two atoms. The master equation now can be solved by defining three Super operators

$$\begin{aligned}&\hat{J}_i^{-}\hat{\rho}=\hat{\sigma}_i^{-}\hat{\rho}\hat{\sigma}_i^{+}\\&\hat{J}_i^{+}\hat{\rho}=\hat{\sigma}_i^{+}\hat{\rho}\hat{\sigma}_i^{-}\\&\hat{J}_i^{3}\hat{\rho}=(\hat{\sigma}_i^{+}\hat{\sigma}_i^{-}\hat{\rho}+\hat{\rho}\hat{\sigma}_i^{+}\hat{\sigma}_i^{-}-\hat{\rho}),\end{aligned} \tag{2}$$

such that J_i^{+}, $\hat{J}_i^{-}$ and $\hat{J}_i^{3}$ with $i=1,2$ obey the commutation relations:

$$\begin{aligned}&\left[\hat{J}_i^{+},\hat{J}_i^{-}\right]\hat{\rho}=\hat{J}_i^{3}\hat{\rho}\\&\left[\hat{J}_i^{3},\hat{J}_i^{\pm}\right]\hat{\rho}=\pm2\hat{J}_i^{\pm}\hat{\rho}.\end{aligned} \tag{3}$$

So we can rewrite Eq. (1) as:

$$d\hat{\rho}/dt=\sum_{i=1}^{2}\Big[2\gamma(n)\hat{J}_i^{+}+2\gamma(n+1)\hat{J}_i^{-}-\gamma\hat{J}_i^{3}-\gamma(2n+1)\Big]\hat{\rho}(t), \tag{4}$$

The formal solution of Eq. (4) can then be written as:

$$\hat{\rho}(t)=\exp\Big[\sum_{i=1}^{2}(2\gamma(n)\hat{J}_i^{+}+2\gamma(n+1)\hat{J}_i^{-}-\gamma\hat{J}_i^{3}-\gamma(2n+1))t\Big]\hat{\rho}(0). \tag{5}$$

The problem is now how to factorize the exponential of super operators [5]. To this purpose, we propose an ansatz as in [4]

$$\begin{aligned}\hat{\rho}(t)=&\exp[(f^{+}(t)\hat{J}_2^{+}]\exp[f^{3}(t)\hat{J}_2^{3}]\\&\times\exp[f^{-}(t)\hat{J}_2^{-}]\exp[(f^{+}(t)\hat{J}_1^{+}]\\&\times\exp[f^{3}(t)\hat{J}_1^{3}]\exp[f^{-}(t)\hat{J}_1^{-}]\\&\times\exp[-2\gamma(2n+1))t]\hat{\rho}(0).\end{aligned} \tag{6}$$

By differentiating both sides of Eq. (6) and using commutation relations (3) as well as Baker-Campbell-Hausdorff formula we find a relation for $d\hat{\rho}/dt$.

Then by comparing the result of it with Eq. (4), we obtain a system of differential equations for time-dependent functions $f^{j}(t)$, for $j=\pm,3$ [6]. With the initial conditions $f^{j}(0)=0$ for $j=\pm,3$, we first solve $\dot{f}^{+}(t)$

which is a nonlinear Riccati equation and then insert it into $\dot{f}^-(t)$, $\dot{f}^3(t)$ to find as:

$$
\begin{aligned}
f^+(t) &= \frac{n(1-\exp(-2\gamma(2n+1)t))}{1+n(1+\exp(-2\gamma(2n+1)t))},\\
f^-(t) &= \frac{(n+1)(1-\exp(-2\gamma(2n+1)t))}{1+n(1+\exp(-2\gamma(2n+1)t))},\\
f^3(t) &= (2n+1)\gamma t\\
&\quad +\ln\frac{(2n+1)\exp(-2\gamma(2n+1)t)}{1+n(1+\exp(-2\gamma(2n+1)t))}.
\end{aligned}
\tag{7}
$$

Once we have found $f^{\pm}(t)$ and $f^3(t)$, we have found the solution to the Eq. (6) and we are ready to apply it to any initial state. In Sec. III we find a practical form of the master equation by using the relations with $i=1,2$

$$
\begin{aligned}
&\hat{J}_i^3\hat{J}_i^3(\hat{J}_i^3\hat{\rho}) = \hat{J}_i^3\hat{\rho}\\
&\hat{J}_i^3(\hat{J}_i^3\hat{\rho}) = 2\hat{J}_i^+(\hat{J}_i^-\hat{\rho}) - \hat{J}_i^3\hat{\rho}\\
&\hat{J}_i^{\pm}(\hat{J}_i^{\pm}\hat{\rho}) = 0, \quad \hat{J}_i^3(\hat{J}_i^{\pm}\hat{\rho}) = \pm\hat{J}_i^{\pm}\hat{\rho},
\end{aligned}
\tag{8}
$$

and exponential expansions:

$$
\exp[(f^{\pm}(t)\hat{J}_i^{\pm}]\hat{\rho} = (1+f^{\pm}(t)\hat{J}_i^{\pm})\hat{\rho} \tag{9a}
$$

$$
\begin{aligned}
\exp[(f^3(t)\hat{J}_i^3]\hat{\rho} = \{&1+(1-\exp[-f^3(t)])\hat{J}_i^3\\
&+2(Cosh(f^3(t))-1)\hat{J}_i^+\hat{J}_i^-\}\hat{\rho}.
\end{aligned}
\tag{9b}
$$

These relations are derived in the appendix.

III. DENSITY MATRIX

Substituting from relations (7) into Eq. (6) and applying the exponential expansions (9) together with relations (8) into Eq. (6), we can finally obtain solution for density matrix as:

$$
\begin{aligned}
\hat{\rho}(t) = [&A^-(t)\,(\hat{J}_1^- + \hat{J}_2^-) + A^+(t)\,(\hat{J}_1^+ + \hat{J}_2^+) + G^-(t)\\
&\times(\hat{J}_2^-\hat{J}_1^+\hat{J}_1^- + \hat{J}_2^+\hat{J}_2^-\hat{J}_1^-) + G^+(t)\,(\hat{J}_2^+\hat{J}_1^+\hat{J}_1^- + \hat{J}_2^+\hat{J}_2^-\hat{J}_1^+)\\
&+B^-(t)\,(\hat{J}_2^-\hat{J}_1^-) + B^+(t)\,(\hat{J}_2^+\hat{J}_1^+) + I(t)\,(\hat{J}_2^+\hat{J}_2^-\hat{J}_1^+\hat{J}_1^-)\\
&+D^-(t)(\hat{J}_2^3\hat{J}_1^- + \hat{J}_2^-\hat{J}_1^3) + D^+(t)(\hat{J}_2^3\hat{J}_1^+ + \hat{J}_2^+\hat{J}_1^3) + F(t)\\
&\times(\hat{J}_2^3\hat{J}_1^+\hat{J}_1^- + \hat{J}_2^+\hat{J}_2^-\hat{J}_1^3) + E^2(t)(\hat{J}_2^3\hat{J}_1^3) + E(t)(\hat{J}_1^3 + \hat{J}_2^3)\\
&+H(t)\,(\hat{J}_1^+\hat{J}_1^- + \hat{J}_2^+\hat{J}_2^-) + C(t)\,\times(\hat{J}_2^-\hat{J}_1^+ + \hat{J}_2^+\hat{J}_1^-) + 1]\\
&\times\exp[-2\gamma(2n+1))t]\hat{\rho}(0),
\end{aligned}
\tag{10}
$$

where time-dependent functions in Eq. (10) are defined as:

$$
\begin{aligned}
&E(t) = (1-\mathrm{e}^{-f^3(t)})\\
&A^{\pm}(t) = f^{\pm}(t)\,\mathrm{e}^{-f^3(t)}\\
&B^{\pm}(t) = (f^{\pm}(t)\,\mathrm{e}^{-f^3(t)})^2\\
&C(t) = f^-(t)f^+(t)\,\mathrm{e}^{-2f^3(t)}\\
&D^{\pm}(t) = f^{\pm}(t)(\mathrm{e}^{-f^3(t)} - \mathrm{e}^{-2f^3(t)})\\
&I(t) = (\mathrm{e}^{+f^3(t)} + (f^-(t)f^+(t)-1)\,\mathrm{e}^{-f^3(t)} - 2)^2\\
&G^{\pm}(t) = f^{\pm}(t)((f^-(t)f^+(t)-1)\,\mathrm{e}^{-2f^3(t)} - 1)\\
&H(t) = (\mathrm{e}^{+f^3} + (f^-(t)f^+(t)-1)\,\mathrm{e}^{-f^3(t)} - 2)\\
&F(t) = (\mathrm{e}^{+f^3} + (f^-(t)f^+(t)-1)\,\mathrm{e}^{-f^3(t)} - 2)\\
&\qquad\times(1-\mathrm{e}^{-f^3(t)}).
\end{aligned}
\tag{11}
$$

Matrix representations of $\hat{J}_i^+$, $\hat{J}_i^-$ and $\hat{J}_i^3$ with $i=1,2$, are determined based on the Pauli matrices in the computational basis (basis of the four dimensional Hilbert space of the two qubits). Pauli matrices in computational basis are given by:

$$
\begin{aligned}
\sigma_1^+ &= \sigma_1^+ \otimes I_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
\sigma_2^+ &= I_1 \otimes \sigma_2^+ = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\\
\sigma_1^- &= \sigma_1^- \otimes I_2 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
\sigma_2^- &= I_1 \otimes \sigma_2^- = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.
\end{aligned}
\tag{12}
$$

By making use of matrix representations of Super operators $\hat{J}_i^{\pm}$, $\hat{J}_i^3$, we can easily obtain matrix representations of other Super-operators in Eq. (10) , most of them have only one, two or four non-zero matrix elements. So the procedure presented in this section does not take into account at all the set of the equations among the matrix elements. Therefore, simplification has been achieved through the present approach. Based on Eq. (10) density matrix elements are given by

$$\rho_{14}(t)=\rho_{14}(0)\exp[-4\gamma(n+1/2)t] \tag{13}$$

$$\rho_{23}(t)=\rho_{23}(0)\exp[-4\gamma(n+1/2)t] \tag{14}$$

$$\rho_{13}(t)=\frac{1}{2n+1}\times\{n(\rho_{13}(0)+\rho_{24}(0))\exp[-\gamma(2n+1)t] +((n+1)\,\rho_{13}(0)-n\rho_{24}(0))\exp[-3\gamma(2n+1)t]\} \tag{15}$$

$$\rho_{12}(t)=\frac{1}{2n+1}\times\{n(\rho_{12}(0)+\rho_{34}(0))\exp[-\gamma(2n+1)t] +((n+1)\,\rho_{12}(0)-n\rho_{34}(0))\exp[-3\gamma(2n+1)t]\} \tag{16}$$

$$\rho_{24}(t)=\frac{1}{2n+1}\times\{(n+1)(\rho_{13}(0)+\rho_{24}(0))\exp[-\gamma(2n+1)t] +[n\rho_{24}(0)-(n+1)\,\rho_{13}(0)]\exp[-3\gamma(2n+1)t]\} \tag{17}$$

$$\rho_{34}(t)=\frac{1}{2n+1}\times\{(n+1)(\rho_{12}(0)+\rho_{34}(0))\exp[-\gamma(2n+1)t]+ +(n\,\rho_{34}(0)-(n+1)\rho_{12}(0))\exp[-3\gamma(2n+1)t]\} \tag{18}$$

$$\rho_{11}(t)=\frac{1}{(2n+1)^2}\times\{n^2+[2n(n+1)\rho_{11}(0)+n(\rho_{33}(0) +\rho_{22}(0))-2n^2\rho_{44}(0)]\exp[-2\gamma(2n+1)t] +[(n+1)^2\rho_{11}(0)-n\rho_{33}(0)-n((n+1)\rho_{22}(0) +n(\rho_{33}(0)-\rho_{44}(0))]\times\exp[-4\gamma(2n+1)t]\} \tag{19}$$

$$\rho_{22}(t)=\frac{1}{(2n+1)^2}\times\{n(n+1) +[(n+1)\rho_{11}(0)-n\rho_{44}(0)-2n(n+1)\rho_{33}(0) +((n+1)^2+n^2)\rho_{22}(0)]\exp[-2\gamma(2n+1)t] +[-(n+1)^2\rho_{11}(0)+n((n+1)\rho_{22}(0)+n(\rho_{33}(0) -\rho_{44}(0)))+n\rho_{33}(0)]\times\exp[-4\gamma(2n+1)t]\} \tag{20}$$

$$\rho_{33}(t)=\frac{1}{(2n+1)^2}\{n(n+1) +[(n+1)\rho_{11}(0)-n\rho_{44}(0)-2n(n+1)\rho_{22}(0) +((n+1)^2+n^2)\rho_{33}(0)]\exp[-2\gamma(2n+1)t] +[-(n+1)^2\rho_{11}(0)+n((n+1)\rho_{22}(0)+n(\rho_{33}(0) -\rho_{44}(0)))+n\rho_{33}(0)]\times\exp[-4\gamma(2n+1)t]\} \tag{21}$$

$$\rho_{44}(t)=\frac{1}{(2n+1)^2}\{(n+1)^2 +[-2(n+1)^2\rho_{11}(0)-(n+1)\rho_{33}(0)+(n+1)\rho_{22}(0) +2n(n+1)\rho_{44}(0)]\exp[-2\gamma(2n+1)t] +[(n+1)^2\rho_{11}(0)-n\rho_{33}(0)-n((n+1)\rho_{22}(0) +n(\rho_{33}(0)-\rho_{44}(0)))]\exp[-4\gamma(2n+1)t]\} \tag{22}$$

IV. CONCURRENCE DYNAMICS

The entanglement of a general mixed state is best identified by examining the concurrence [7]–[8], an entanglement measure that relates to the density matrix of the system $\hat{\rho}$. Note that concurrence varies from C=0 for a separable state to C=1 for a maximally entangled state. The concurrence for a two qubit system is defined as:

$$C(t)=\max\{0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\} \tag{23}$$

where the λ_i (i=1,2,3,4) characterizes the eigenvalues of the non-hermitian matrix $\rho(t)\tilde{\rho}(t)$ arranged in decreasing order of magnitude and $\tilde{\rho}(t)$ is defined as:

$$\tilde{\rho}(t)=(\sigma_y^{(1)}\otimes\sigma_y^{(2)})\rho^*(t)(\sigma_y^{(1)}\otimes\sigma_y^{(2)}), \tag{24}$$

here we consider the initially entangled qubits to be in a mixed state [9] given by the density matrix

$$\begin{aligned}&\hat{\rho}(0)=(a|1\rangle\langle 1|+d\,|4\rangle\langle 4|+(b+c)|\psi\rangle\langle\psi|)/3\\ &|\psi\rangle=(\sqrt{b}\,|2\rangle+e^{i\chi}\sqrt{c}\,|3\rangle)/\sqrt{b+c}\\ &(a+b+c+d)/3=1.\end{aligned} \tag{25}$$

Note that the entanglement part of the state depends on the initial phase χ. In the above relations, a, b and c are parameters governing the nature of the initial state of the two entangled qubits and $z = e^{i\chi}\sqrt{bc}$ is the single photon coherence. In the matrix form, $\hat{\rho}(0)$ is then given by:

$$\hat{\rho}(0) = \frac{1}{3}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{26}$$

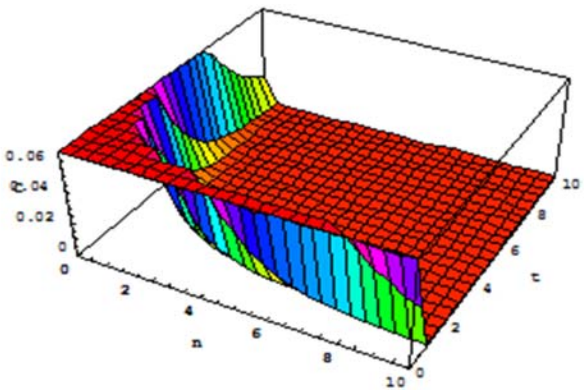

Fig. 1. Concurrence vs. photon number and time for two qubits with initial conditions $\gamma = 0.18$, $\chi = 0$, $b = c = 1$, $d = 0.6$, and a=0.4.

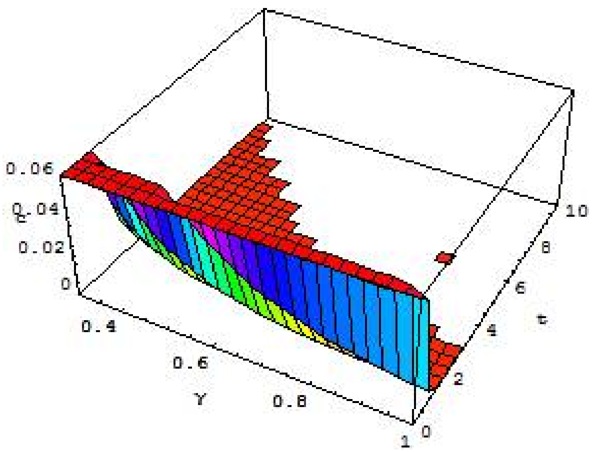

Fig. 2. Concurrence Vs. spontaneous emission and time for two qubits with initial conditions.

We consider a particular class of mixed states with a single parameter a satisfying $a \geq 0$, and $b = c = |z| = 1,\ d = 1 - a$. In Fig. 1, we have plotted the concurrence versus the photon number and time for a mentioned mixed state introduced by (26). For a mixed state, as it is seen, concurrence is lower than its maximal value at $t = 0\ s$ and for the larger value of n, the amplitude of the concurrence decreases more rapidly with the time.

Figure 2 describes the behavior of concurrence as a function of spontaneous emission and time. One finds that the larger the spontaneous emission is, the more quickly the amplitude of the concurrence is suppressed.

V. Conclusion

To summarize, we solved master equation for two non-interacting entangled qubits in contact with thermal bath at finite temperature by using super operator techniques.

Through this procedure, we derived density matrix in a compact form without resort to writing it down as a set of equations among the matrix elements. So the solution is directly obtained in an operator form. Furthermore, we needed to solve less differential equation than for the matrix elements.

Then we adopted one entanglement measure to investigate dynamics of the system. As expected, we obtained analytical evidence that the thermal photons and also spontaneous emission of atoms were responsible for dissipation of entanglement of system.

Appendix A

Derivation of Eq. (9a) and Eq. (9b)

By using the series expansion of exponential term in Eq. (9a)

$$\exp[(f^{\pm}(t)\hat{J}_i^{\pm}]\hat{\rho} = \Big(1 + (f^{\pm}(t)\hat{J}_i^{\pm})^1/1! + (f^{\pm}(t)\hat{J}_i^{\pm})^2/2! + (f^{\pm}(t)\hat{J}_i^{\pm})^3/3! + \ldots\Big)\hat{\rho} \tag{A1}$$

and applying the relations (8) to each term of (A1), it is seen that only the first two terms remain

$$\exp[(f^{\pm}(t)\hat{J}_i^{\pm}]\hat{\rho} = (1 + f^{\pm}(t)\hat{J}_i^{\pm})\hat{\rho}. \tag{A2}$$

To derive Eq. (9b), we apply relations

$$\hat{J}_i^3 \hat{J}_i^3 (\hat{J}_i^3 \hat{\rho}) = \hat{J}_i^3 \hat{\rho}$$
$$\hat{J}_i^3 (\hat{J}_i^3 \hat{\rho}) = 2\hat{J}_i^+ (\hat{J}_i^- \hat{\rho}) - \hat{J}_i^3 \hat{\rho}, \quad \text{(A3)}$$

into the series expansion of Eq. (9b)

$$\exp[(f^3(t)\hat{J}_i^3]\hat{\rho} = \sum_{n=0}^{\infty}[(f^3(t)\hat{J}_i^3)^{2n}\hat{\rho}/(2n)! + \sum_{n=0}^{\infty}[(f^3(t)\hat{J}_i^3)^{2n+1}\hat{\rho}]/(2n+1)! \quad \text{(A4)}$$

and after some calculations we derive Eq. (9b) as below,

$$\begin{aligned} &\exp[(f^3(t)\hat{J}_i^3]\hat{\rho} = 1 + \sum_{n=1}^{\infty}(f^3(t))^{2n}/(2n)! \\ &\times(2\hat{J}_i^+(\hat{J}_i^-\hat{\rho}) - \hat{J}_i^3\hat{\rho}) + \sum_{n=0}^{\infty}(f^3(t))^{2n+1}/(2n+1)! \\ &\times\hat{J}_i^3\hat{\rho} = 1 + (f^3(t))^2[(2\hat{J}_i^+(\hat{J}_i^-\hat{\rho}) - \hat{J}_i^3\hat{\rho})]/2! + (f^3(t))^4 \\ &\times[(2\hat{J}_i^+(\hat{J}_i^-\hat{\rho}) - \hat{J}_i^3\hat{\rho})]/4! + ... + Sinh(f^3(t))\hat{J}_i^3\hat{\rho} = 1 \\ &+[Coshf^3(t)) - 1](2\hat{J}_i^+(\hat{J}_i^-\hat{\rho}) - \hat{J}_i^3\hat{\rho}) + Sinh(f^3(t))\hat{J}_i^3\hat{\rho} \\ &= 1 + [2Cosh\ (f^3(t)) - 2]\hat{J}_i^+(\hat{J}_i^-\hat{\rho}) + [1 + Sinh(f^3(t)) \\ &- Cosh(f^3(t))]\hat{J}_i^3\hat{\rho} = 1 + 2[Cosh\ (f^3(t)) - 1]\hat{J}_i^+(\hat{J}_i^-\hat{\rho}) \\ &+[1 - \exp(-f^3(t))]\hat{J}_i^3\hat{\rho}. \end{aligned} \quad \text{(A5)}$$

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